Public and Catholic District School Board Writing Partnerships

Course Profile

Functions and Relations

Grade 11
University Preparation
MCR3U

• *for teachers by teachers*

This sample course of study was prepared for teachers to use in meeting local classroom needs, as appropriate. This is not a mandated approach to the teaching of the course. It may be used in its entirety, in part, or adapted.

Spring 2001

Acknowledgments

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Course Overview
Functions and Relations, Grade 11, University Preparation, MCR3U

Prerequisite: Principles of Mathematics, Grade 10, Academic

Course Description

This course introduces some financial applications of mathematics, extends students' experiences with functions and trigonometry, and introduces second-degree relations. Many of the expectations of this course are based on direct extensions of concepts introduced in Grades 9 and 10. Having previously explored linear and quadratic relationships, students study polynomial and rational functions, and investigate the relationship between functions and their inverses. Students continue their study of trigonometry and discover new properties and contexts to which it can be applied. Graphing and algebraic skills are also consolidated and extended in this course. Identifying connections between the algebraic and graphic representations of functions continues to be an important skill.

Successful completion of MCR3U Functions and Relations will prepare students for any of the five Grade 12 University and College Preparation courses, and provides the necessary foundation for mathematically rich university programs. In particular, this course should be taken by students who are planning to study engineering, computer science, pure mathematics, or the physical sciences at the university level. This course shares a core set of expectations with the Grade 11 Functions course, MCF3M, which is comprised of three strands: Financial Applications of Sequences and Series, Trigonometric Functions, and Tools for Operating and Communicating with Functions. Completion of either of these two Grade 11 programs will prepare students for the Grade 12 courses Advanced Functions and Introductory Calculus (MCB4U), Mathematics of Data Management (MDM4U), Mathematics for College Technology (MCT4C), and College and Apprenticeship Mathematics (MAP4C). In addition to this common core, the Functions and Relations course contains some extension to the core plus a fourth strand, Investigations of Loci and Conics, which provides students with the knowledge and skills necessary for MGA4U, Geometry and Discrete Mathematics.

In addition to the Investigations of Loci and Conics strand and operations on complex numbers, the MCR3U course differs from the MCF3M course in timing and context. The pace of delivery in MCR3U will require students to consistently demonstrate the ability to:

- investigate and construct mathematical concepts independently;
- conjecture and, through inquiry, test a hypothesis;
- generate multiple types of solutions to complex problems which may cross strands, require the use of appropriate technology, and require abstract thinking (e.g., the consideration of cases);
- expand the depth of their inquiry in order to solve higher-order problems;
- analyse and design proofs from multiple perspectives.

Because of the destination intended for students enrolled in MCR3U, the contextual examples and activities should be drawn largely from the areas of engineering, the physical sciences, computer science, and pure mathematics.

In the Financial Applications of Sequences and Series strand, students will acquire the tools required to make sound personal financial decisions. Students will investigate and solve problems involving applications of sequences related to compound interest, annuities, and financial decision-making. In the Trigonometric Functions strand, students will investigate and apply properties of the primary trigonometric functions and develop a competency in the manipulation of these functions. This strand has been divided into two units: Trigonometry (covering radian measure and the sine and cosine laws) and Trigonometric Functions (with particular emphasis on the study of sinusoidal functions). The Tools for Operating and Communicating with Functions strand allows students to develop skills in operating with various algebraic expressions and to develop facility in using function notation and in communicating reasoning. This strand has also been divided into two units: Exploring Functions: Connecting Algebra and Geometry and Function Notation, Inverses, and Transformations. In the Investigations of Loci and Conics strand, students will extend their inquiry of loci into a study of conics. Students will become proficient in two- and three-dimensional modelling, which is essential to achieving success in MGA4U, Geometry and Discrete Mathematics.

How This Course Supports the Ontario Catholic School Graduate Expectations

This course encourages the Catholic learner to develop his/her God-given gifts and abilities to promote growth toward personal responsibility in preparation for a chosen career path. Throughout this course, emphasis should be placed on moral, ethical, and realistic decision-making in an effort to build responsible citizenship. The classroom environment should instil a spirit of cooperation, rather than competition amongst students, and should foster a collaborative sense of community. This course provides many opportunities for students to work effectively as interdependent team members and to acknowledge others for their opinions.

Course Notes

This course profile builds on previous mathematics course profiles written for Grades 9 and 10. The Grade 11 course profiles produced by the Catholic and Public systems represent a collaborative effort between the two writing teams. While not as detailed as the Grade 9 and 10 profiles, each is designed to complement and supplement the other. Due to the common core of learning expectations between the MCR3U and MCF3M courses, a common unit breakdown has been suggested, and four different sample units have been developed. Thus, the MCR3U and MCF3M course profiles can also be used as complementary resources. With appropriate adjustments to the complexity of problems and the need for abstract thinking, activities from either profile can be used by a teacher of either course. It should be noted that in this course, it is appropriate in certain situations for mathematics itself to provide the context for new concepts, while in MCF3M it would be more appropriate to provide other contexts as well. The sample units provided in this (MCR3U) profile are Exploring Functions: Connecting Algebra and Geometry and Trigonometric Functions, while the sample units provided in the MCF3M profile are Function Notation, Inverses, and Transformations and Financial Applications of Sequences and Series. In addition to these four complete "sample" units, a less-detailed Unit Overview chart offers a recommended clustering of expectations for each of the remaining units, providing a starting point from which teachers can develop their own, individualized units.

For some students, mathematics is perceived to be a collection of isolated and complex topics, each requiring skills that may soon be forgotten. The mathematics teacher must address these perceptions by creating a context in which students can learn and connect concepts and skills. Students must be exposed to a variety of teaching, learning, and problem solving techniques to best synthesize the information presented by the curriculum, and should be provided applications and context to bring meaning to their learning.

The activities in this profile are designed to both introduce and consolidate skills necessary for success in this course. These activities can be used in conjunction with or independently of one another. Alternate teaching strategies and suggestions for technological tools are included to help teachers present the lessons contained in the activities.

Because this course has been designed to prepare students for further mathematical studies at university, the specific nature of the learning activities should reflect this destination. In particular, students in this course should routinely be challenged with investigations and problems which require sustained, independent effort. Students destined for university should have the opportunity to develop and demonstrate a high level of complex problem-solving ability.

Students with learning disabilities will need specific guidance in order to benefit from the investigative approach presented in this profile. Review of prerequisite skills and instructions in the use of technology, and in particular graphing calculators, will be required before any activities are begun. Clear and precise instructions with examples will need to be provided.

Several of the activities presented in this profile include extensions of the required content, which can be used to meet the need to challenge gifted students. Other accommodations may include allowing for student preferences in supplemental learning, altering the pace of instruction, creating a flexible classroom environment, and using specific instructional strategies. Creative approaches to problem solving must be encouraged.

The Achievement Chart for Mathematics is the basis of all assessment and evaluation for this course. The Principles of Mathematics – Academic (Grade 10) Public Course Profile includes charts suggesting strategies that can be used for the assessment and evaluation of all categories of the Achievement Chart (p. 11). In addition, a chart outlining the component actions that are needed for successful inquiry and problem solving in particular is also included in their profile (p. 12). These charts provide an excellent base with which to begin the implementation of these strategies, and for teachers of this course to extend, depending on their degree of readiness. Another excellent resource is the *Concerning Assessment and Reflective Evaluation* (CARE) package of materials, available for free download at http://www.oame.on.ca. Among the resources included in this package are generic rubrics for Communication and Thinking/Inquiry/Problem Solving skills, along with suggested applications of these instruments.

Units: Titles and Time

* Unit 1	Exploring Functions: Connecting Algebra and Geometry	18 hours
Unit 2	Function Notation, Inverses, and Transformations	18 hours
Unit 3	Trigonometry	13 hours
* Unit 4	Trigonometric Functions	19 hours
Unit 5	Financial Applications of Sequences and Series	23 hours
Unit 6	Investigations of Loci and Conics	13 hours
Unit 7	Final Summative Assessment	6 hours

* These units are fully developed in this Course Profile.

Unit Overviews

Unit 1: Exploring Functions: Connecting Algebra and Geometry

Time: 18 hours

Ontario Catholic School Graduate Expectations: CGE2b, CGE2c, CGE3c, CGE3e, CGE4a, CGE4b, CGE4f, CGE5a, CGE5g, CGE7b, CGE7j.

Unit Description
Students investigate quadratic functions and related concepts from algebraic and geometric perspectives, in order to deepen their understanding and prepare them for further explorations of functions and relations. A winter recreation theme is loosely woven throughout selected activities in the unit, providing a contextual framework for students to solve problems, both with and without the use of graphing technology. Students solve first-degree inequalities and graph their solutions on number lines. Skills involving operations with polynomials and rational expressions are consolidated, and then extended to the complex number system, which is introduced in this unit. Students apply the method of completing the square in order to solve maximum/minimum problems involving quadratic functions. Algebraic and graphical methods are used to determine the roots of quadratic equations. The exponent laws are applied to expressions which have powers containing integer and rational exponents. Students discover the nature of exponential functions and solve exponential equations.

Unit Overview Chart

Activity	Time	Expectations	Assessment	Tasks
1. A Range of Possibilities	75 min	OCV.01, OCV.03 OC1.01, OC1.02, OC3.04, OC3.05	Knowledge Communication	Solve and graph inequalities; Perform operations with polynomials
2. Ski-Jumping to the Max!	150 min	OCV.01, OCV.03 OC1.03, OC1.05, OC3.01, OC3.05	Knowledge Inquiry Communication Application	Complete the square; Investigate the graphs of quadratic functions
3. Rooting Around the Parabola	150 min	OCV.01, OCV.03 OC1.04, OC1.05, OC3.02, OC3.05	Knowledge Inquiry Communication Application	Determine real/complex roots of quadratic equations; Relate roots to *x*-intercepts of quadratic functions
4. Complex Basics are Basically not Complex!	150 min	OCV.01, OCV.03 OC1.06, OC3.04, OC3.05	Knowledge Inquiry Communication	Perform algebraic operations on complex numbers; Graph numbers on the complex plane
5. Can We Please be Rational?!	150 min	OCV.01, OCV.03 OC1.07, OC3.01, OC3.03, OC3.04, OC3.05	Knowledge Communication	Perform algebraic operations on rational expressions; State restrictions on the variables
6. Power Play	75 min	OCV.01, OCV.03 OC1.08, OC3.01, OC3.05	Inquiry Communication Application	Explore powers with rational exponents; Apply exponent laws to powers containing integer and rational exponents
7. It's Snowing Cats and Dogs!	150 min	OCV.01, OCV.03 OC1.09, OC.3.03, OC3.05	Knowledge Application	Explore exponential relations; Solve exponential equations
8. Summative Assessment	75 min	All expectations within unit	Knowledge Inquiry Communication Application	Summative assessment

Any additional time can be allocated for remediation and consolidation of skills at the discretion of the teacher, depending on the needs of the students.

Unit 2: Function Notation, Inverses, and Transformations

Time: 18 hours

Unit Description

Through authentic models, students are introduced to the definition of a function and the notations associated with it. Students use graphing technology and paper-and-pencil tasks to investigate the properties of functions and their inverses, and the transformations of functions. The investigations are used to introduce and extend the use of function notation to inverses and transformations. Students explore the domain and range of functions, inverses, and transformations.

Unit Overview Chart

Activity	Time	Expectations	Assessment	Tasks
1. Wrap Around Follow-Up Skills	150 min 75 min	OCV.02, OCV.03 OC2.01, OC2.02, OC3.01, OC3.02, OC3.04 CGE2c, CGE3c, CGE4f, CGE5e, CGE5g	Knowledge Inquiry Communication Application	Investigate the concept of a function; Formalize the definition of the term function; Apply function notation
2. Home on the Range Follow-Up Skills	150 min 75 min	OCV.02, OCV.03 OC2.02, OC2.03, OC3.03, OC3.05 CGE4f, CGE5a	Knowledge Communication Application	Explore through the use of graphing technology the properties of various functions; Investigate domain and range
3. Follow the Bouncing Ball	105 min	OCV.02, OCV.03 OC2.02, OC3.03, OC3.04, OC3.05 CGE2b, CGE3c, CGE5a	Knowledge Communication Application	Develop a model with the use of graphing technology; Apply appropriate function notation with the model; Investigate properties of the function/model
4. Let's Switch Seats! Follow-Up Skills	105 min 75 min	OCV.02, OCV.03 OC2.04, OC3.03, OC3.05 CGE3c, CGE4f	Inquiry Application	Investigate the properties of inverse functions; Discover algebraic approaches for finding the inverse of functions
5. On the Move	150 min	OCV.02, OCV.03 OC2.06, OC2.07, OC2.08, OC3.02, OC3.04 CGE2c, CGE5a	Knowledge Communication Application	Investigate the effect of transformations on mathematical functions; Apply appropriate function notation to transformations of functions
6. Be my Valentine	105 min	OCV.02, OCV.03 OC2.06, OC2.07, OC2.08, OC3.01, OC3.05 CGE2b, CGE2f	Knowledge Inquiry Communication Application	Apply and analyse transformations of functions

7. Consolidating and Connecting	105 min	OCV.02, OCV.03 OC1.03, OC2.02, OC2.06, OC2.07, OC3.02, OC3.03 CGE3c, CGE4f, CGE5a	Knowledge Inquiry Communication Application	Summative assessment

Any additional time can be allocated for remediation and consolidation of skills at the discretion of the teacher, depending on the needs of the students.

Unit 3: Trigonometry

Time: 13 hours

Ontario Catholic School Graduate Expectations: CGE2b, CGE2c, CGE3c, CGE3e, CGE4a, CGE4b, CGE4f, CGE5a, CGE5g, CGE7b, CGE7j.

Unit Description

Students consolidate and extend concepts first introduced in Grade 10. Students use the primary trigonometric ratios, the sine law, and the cosine law to model and solve two- and three-dimensional problems involving acute, right, and oblique triangles. Students investigate the relationship between degree and radian measure, and explore the use of the unit circle and special triangles to determine selected values of the primary trigonometric ratios. Methods of proof are introduced and applied to verify trigonometric identities. Students develop the skills to manipulate and solve trigonometric equations.

Unit Overview Chart

Cluster	Expectations	Assessment	Focus
1	TFV.02 TF2.01, TF2.02, TF2.03, TF2.07	Knowledge Inquiry Application	Define terms and concepts; Convert degrees to radians; Apply radian measure
2	TFV.01, TF1.02, TF2.03	Knowledge Inquiry Communication Application	Review the primary trigonometric ratios; Review the sine and cosine laws for acute triangles; Explore the sine and cosine laws for oblique triangles; Investigate the ambiguous case of the sine law
3	TFV.02 TF2.03, TF2.04	Knowledge Inquiry Communication Application	Investigate special triangles
4	TFV.02 TF1.01, TF2.05	Knowledge Inquiry Communication Application	Derive the unit circle; Use the Pythagorean theorem to prove identities; Discuss the use of proof
5	TFV.02 TF2.03, TF2.06, TF2.07	Knowledge Inquiry	Solve linear and quadratic trigonometric equations
6	All expectations within unit	Knowledge Inquiry Communication Application	Summative assessment

Unit 4: Trigonometric Functions

Time: 19 hours

Unit Description

Students investigate the periodic nature and graphical properties of the primary trigonometric functions. Using technology, students explore the effects of simple transformations on their graphs and equations. Students apply these concepts to model authentic problems.

Unit Overview Chart

Activity	Time	Expectations	Assessment	Tasks
1. Surf's Up! Let's Catch the Sine Wave	75 min	TFV.02, TFV.03 TF2.07, TF3.01 CGE3c	Knowledge	Derive $y = \sin x$ and $y = \cos x$ from the unit circle
2. Transformations: More than Meets the Eye Follow-Up Skills Give Me a Sine Follow-Up Skills	75 min 30 min 75 min 45 min	TFV.02, TFV.03, OCV.02 TF2.07, TF3.01, TF3.02, TF3.03, TF3.04, TF3.05, OC2.06, OC2.07, OC2.08 CGE3c, CGE5a, CGE5e, CGE5f	Knowledge Inquiry Communication	Investigate the properties of various transformations of $y = \sin x$ and $y = \cos x$
3. Don't Go Off on a Tangent Follow-Up Skills	75 min 30 min	TFV.03 TF3.01, TF3.06 CGE3c, CGE 4b, CGE 4f	Knowledge Inquiry Communication	Investigate the properties of $y = \tan x$
4a. It's a Spring Thing	150 min	TFV.03, TFV.04, OCV.02 TF3.02, TF3.03, TF3.04, TF3.05, TF4.01, TF4.02, OC2.06, OC2.08 CGE2c, CGE3c, CGE4b, CGE5a, CGE5g	Knowledge Inquiry Communication Application	Model the motion of a mass on a spring
4b. Ferris Fair Follow-Up Skills	150 min 75 min			Model the height of a gondola on a Ferris wheel
4c. Let the Sine Shine In Follow-Up Skills	150 min 60 min			Model an hours of daylight function
5. Summative Assessment	150 min	All expectations within unit CGE2b, CGE3c, CGE3e	Knowledge Inquiry Communication Application	Apply concepts in familiar and unfamiliar contexts

Unit 5: Financial Applications of Sequences and Series

Time: 23 hours

Ontario Catholic School Graduate Expectations: CGE2b, CGE2c, CGE3b, CGE3c, CGE3e, CGE4a, CGE4f, CGE5a, CGE5c, CGE7b, CGE7c.

Unit Description
Students investigate arithmetic and geometric sequences and series. This knowledge serves as the basis for applications of personal finance. Students develop the formula for compound interest and solve problems related to compound interest and annuities. As skills are developed, students use spreadsheets to investigate the cost of borrowing when interest rates, compound periods, lending terms, etc., are varied. The activities are designed to reflect the type of decisions that students are likely to face in the future. Students apply skills with linear and exponential functions.

Unit Overview Chart

Activity	Time	Expectations	Assessment	Tasks
1. Investigating Attributes of Sequences	150 min	FAV.01 FA1.01, FA1.02	Inquiry Communication	Investigate sequences
2. Summing Up: Arithmetic Sequences and Series	150 min	FAV.01, FAV.02 FA1.04, FA1.05, FA2.04	Knowledge Inquiry Application	Develop formula for arithmetic sequences and series
3. Compound Interest: Exploring Geometric Sequences	150 min	FAV.01, FAV.02 FA1.03, FA1.04, FA2.01, FA2.02, FA2.05	Knowledge Communication Application	Use a financial application to investigate geometric sequences; Develop the compound interest formula
4. Applications: Finding the Amount and the Present Value of a Long Term Investment	75 min	FAV.02, FAV.03 FA2.02, FA3.01, FA3.02	Knowledge Application	Extend knowledge of the compound interest formula and introduce financial applications of a graphing calculator
5. Introduction to Geometric Series	75 min	FAV.01 FA1.05	Knowledge Application	Develop the formula for the sum of a geometric series
6. Applications: Finding the Amount and the Present Value of an Annuity.	150 min	FAV.02 FA2.02, FA2.03 FA3.05	Inquiry Communication	Apply geometric sequences and series to finding the amount and present value of an annuity
7. What Happens When?: Changing the Time, Rate and Amount.	150 min	FAV.03 FA3.01 FA3.05	Knowledge Inquiry Communication Application	Use financial applications of a graphing calculator to investigate the effect of changing conditions when borrowing and saving
8. Mortgages: How They Work.	150 min	FAV.03 FA3.02, FA3.03, FA3.04 FA3.05	Communication Application	Use technology to generate amortization tables
9. Financial Decision Making: A Case Study	300 min	All expectations within unit	Knowledge Inquiry Communication Application	Summative assessment

Any additional time can be allocated for remediation and consolidation of skills at the discretion of the teacher, depending on the needs of the students.

Unit 6: Investigations of Loci and Conics

Time: 13 hours

Ontario Catholic School Graduate Expectations: CGE2b, CGE2c, CGE3c, CGE3e, CGE4a, CGE4b, CGE4f, CGE5a, CGE5g, CGE7b, CGE7j.

Unit Description

Students investigate the concept of locus of points, using dynamic geometry software. Conic sections are introduced as second-degree relations, containing some examples that are not functions. Physical models (e.g., string models, sketches) representing conic sections are constructed, which serve to illustrate their geometric properties. By translating to standard position, students identify conic sections by type, given their equations. Students determine the intersection of lines and conics, and solve problems involving the application of conics.

Unit Overview Chart

Cluster	Expectations	Assessment	Focus
1	LCV.01 LC1.01, LC1.02, LC1.03	Knowledge Inquiry	Construct geometric models of loci with and without dynamic geometry software
2	LCV.01, LCV.02 LC1.02, LC1.04, LC1.05, LC2.05	Communication Application	Construct geometric models of conic sections with and without dynamic geometry software
3	LCV.02 LC2.01, LC2.03, LC2.04	Knowledge Communication	Identify geometric properties of conics in standard position
4	LCV.02 LC2.02, LC2.03, LC2.04	Knowledge Communication	Translate conics to standard position; Identify geometric properties of generic conics
5	LCV.02, LCV.03 LC2.04, LC3.03	Knowledge Inquiry	Determine points of intersection of lines and conics
6	LCV.03 LC3.01, LC3.02	Inquiry Application	Investigate applications of conics
7	All expectations within unit	Knowledge Inquiry Communication Application	Summative assessment

Unit 7: Final Summative Assessment

Time: 6 hours

Ontario Catholic School Graduate Expectations: CGE2b, CGE2c, CGE3c, CGE3e, CGE4a, CGE4b, CGE4f, CGE5a, CGE5g, CGE7b, CGE7j.

Unit Description

Summative assessment should be designed to provide the opportunity for students to demonstrate comprehensive learning in each of the four achievement categories. Some ideas are suggested in the chart that follows, however any of the various assessment tools mentioned in the *Assessment Strategies* section could be used. A short paper-and-pencil task would review key terms, skills, and concepts. Investigations comparing the buying and leasing of a car yield a wide variety of applications pertaining to both personal finance and the modelling of functions. An assignment exploring trigonometric inverses (for example, the arcsine function) would serve to review the concepts introduced in the functions and trigonometry units. This topic also provides students with an exposure to a subject further explored in Grade 12. An additional activity linking the conics and trigonometry strands (using the unit circle and parametric equations, perhaps) could also be designed. These topics are suggested as one possible way to revisit the expectations in a new mathematical context. Accordingly, students are to be assessed solely on the expectations of this course, and not on the extension topics themselves. Due to the particular emphasis of cumulative tests and examinations in university and college programs, a formal examination should be play a prominent role in the final summative assessment of the student.

Unit Overview Chart

Cluster	Expectations	Assessment	Focus
1	All strands	Knowledge Application	Review key concepts and terms
2	• Financial Applications of Sequences and Series • Tools for Operating and Communicating with Functions	Knowledge Inquiry Communication Application	Examine financial commitments of owning a car: buying vs. leasing
3	• Trigonometric Functions • Tools for Operating and Communicating with Functions	Knowledge Inquiry Communication Application	Explore trigonometric inverses
4	• Trigonometric Functions • Investigations of Loci and Conics	Knowledge Inquiry Communication Application	Explore parametric equations
5	All strands	Knowledge Inquiry Communication Application	Final examination

Some Considerations for Alternate Delivery Models

Because of the similarity of the learning expectations of the MCR3U and MCF3M courses, it is possible that some schools may run split classes, combining both courses in the same room. While this situation should be avoided wherever possible, timetabling issues may make it impossible to have separate classrooms for each of these two courses. Therefore, in order to satisfy the needs of all students in this situation, the following delivery model, patterned after the scope and sequence of the MCF3M Course Profile, is proposed.

Note that the Investigations of Loci and Conics unit is to be delivered largely as an independent study, throughout the length of the course. Only students enrolled in MCR3U would be required to complete this unit. The teacher should provide instructional support where necessary. It is suggested during these times that the teacher have MCF3M students otherwise engaged in activities designed to consolidate skills in the three core strands.

As the times for the remaining units have all been increased, it is expected that the pace of the delivery of course will be appropriately altered. This will allow MCR3U students time to expand the depth and scope of their inquiry and problem-solving skills relating to relevant subject matter. The Financial Applications of Sequences and Series unit can be delivered either before or after the two trigonometry units, depending on the students' needs.

Units: Titles and Time

* Unit 1	Exploring Functions: Connecting Algebra and Geometry	20 hours
Unit 2	Function Notation, Inverses, and Transformations	20 hours
Unit 3	Financial Applications of Sequences and Series	25 hours
* Unit 4	Trigonometry	13 hours
Unit 5	Trigonometric Functions	22 hours
Unit 6	Investigations of Loci and Conics	❖ 0
Unit 7	Final Summative Assessment	10 hours

* These units are fully developed in this Course Profile.
❖ Independent study unit that must be completed by all MCR3U students.

Teaching/Learning Strategies

In order to address the wide range of expectations in this course, a variety of teaching, learning, and assessment strategies and tools need to be used. Teachers should assume a variety of roles (including guide, facilitator, consultant, and instructor), and should employ a variety of strategies including:

- a balance of whole-class, small group, mixed-ability structured group, and individual instruction through student-centred and teacher-directed activities (group work should be carefully structured along cooperative learning principles to be effective);
- the use of rich contextual problems which engage students and provide them with opportunities to demonstrate learning, and appreciate the need for new skills;
- the prompting, supporting, and challenging of individual students as well as the class as a whole;
- approaches that will accommodate multiple learning styles (e.g., the provision of verbal and written instructions, the inclusion of hands-on activities, etc.);
- the use of technological tools and software (e.g., graphing software, dynamic geometry software, the Internet, spreadsheets, and multimedia) in activities, demonstrations, and investigations to facilitate the exploration and understanding of mathematical concepts;
- the use of learning/performance tasks that are designed to link several expectations and give the students occasion to demonstrate their optimal levels of achievement through the demonstration of skill acquisition, the communication of results, the ability to pose extending questions following an inquiry, and the determination of a solution to unfamiliar problems;
- the use of accommodations, remediation, and/or extension activities, where necessary, to meet the needs of exceptional students;
- the provision of opportunities for students to practise and extend their skills and knowledge outside of the classroom.

In addition to the contribution of the teacher, students themselves should play an active role in their own learning. In order to successfully complete the requirements of this course, students are expected to:

- develop an increased responsibility for their own learning;
- be accountable for prerequisite skills;
- participate as active learners;
- engage in explorations using technology;
- apply individual and group learning skills;
- describe mathematical patterns that emerge verbally, algebraically, and visually in the course of learning.

Assessment Strategies

An effective assessment program in mathematics must include a balance of diagnostic, formative and summative assessment instruments that incorporate the categories of learning as defined in The Achievement Chart for Mathematics. One approach is shown below:

	Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application
final examinations	✓	✓	✓	✓
journals	✓		✓	✓
observations		✓	✓	✓
oral presentations	✓		✓	
performance tasks	✓	✓	✓	✓
portfolios	✓	✓	✓	✓
quizzes	✓			
reports/assignments	✓		✓	✓
student-teacher conferences	✓		✓	
unit tests	✓	✓	✓	✓

Assessment tools such as observational checklists, performance criteria, rubrics, The Achievement Chart for Mathematics, marking schemes, rating scales, peer evaluation, and self-evaluation can and should be used to assist in developing objective and consistent evaluations of student achievement.

Assessment & Evaluation of Student Achievement

Assessment, as defined in the document *Ontario Secondary Schools, Grades 9-12: Program and Diploma Requirements, 1999*, is “the process of gathering information from a variety of sources (including assignments, demonstrations, projects, performances, and tests) that accurately reflects how well students are achieving the curriculum expectations” (p. 31). Assessment tools should be designed to allow students to demonstrate the full extent of their learning across the four categories of knowledge and skills. As teachers will use a variety of assessment tools, it is necessary to ensure that a consistent standard is maintained. These tools should be developed with the learning expectations of the course as the criteria for this standard. Thus, a grade of 70-79% using an objective marking scheme should be equivalent to a Level 3 performance, as defined by the Achievement Chart. Teachers may find it more appropriate to use rubrics to assess Thinking/Inquiry/Problem Solving, and Communication skills, but to use objective scales for Knowledge/Understanding, and Application skills. High-quality assessment can measure individual and group performance, and individual performance within a group.

The students’ effective demonstration of communication skills is an essential component of this course when evaluating achievement. Students are required to display and convey their knowledge and understanding of concepts, share their process of thought and inquiry, and justify their application of concepts in an unfamiliar situation. In addition, their *ability* to communicate these skills is also assessed.

It should also be noted that teachers must continue to expand their understanding of Application skills to include non-routine applications. This view requires a shift from the *specific* application of concepts (i.e., familiar situations), to the *general* application of concepts (i.e., unfamiliar situations).

Assessment strategies and tools must address a wide variety of teaching and learning styles in addition to the criteria established by the learning expectations. Tests consisting only of questions that ask students to perform algorithms and apply their knowledge do not necessarily offer an opportunity for students to demonstrate Level 4 performance.

Also, it is understood that students will meet course expectations at a variety of performance levels. An effective and well-balanced assessment program will provide students with several opportunities to demonstrate growth and improvement over time, across all of the knowledge and skill categories.

Evaluation, as defined by *Ontario Secondary Schools, Grades 9-12: Program and Diploma Requirements, 1999*, is "the process of judging the quality of a student's work on the basis of established achievement criteria, and assigning a value to represent that quality" (p. 31). Whereas assessment is the collection of information about student performance in a variety of methods, evaluation is the determination of a quantitative value describing the student's overall level of achievement. Effective assessment, evaluation, and reporting require the teacher to do more than just average marks. While averaging may be more useful in some Knowledge and Application skill categories, it is not comprehensive enough for accurate reporting in the Inquiry and Communication skill categories. The use of rubrics is a suggested technique for these categories. As students can be expected to improve their performances over time, particular emphasis should be placed on their most recent and most consistent level of achievement.

Students who receive a final performance evaluation of Level 3 or better are well prepared for work in the university preparation courses Geometry and Discrete Mathematics (MGA4U), Advanced Functions and Introductory Calculus (MCB4U), and Mathematics of Data Management (MDM4U). Accordingly, in order to prepare students for the academic reality of most mathematically rich university programs, proper attention should be placed on the students' effective preparation for a comprehensive final examination. While other rich, performance-based activities can and should be part of the Final Summative Assessment unit, a formal examination should play a significant role in this particular course.

Seventy per cent of the grade will be based on assessments and evaluations conducted throughout the course. Thirty per cent of the grade will be based on a final evaluation in the form of an examination, performance, essay, and/or other method of evaluation.

Accommodations

Teachers should refer to the students' Individual Education Plans (IEP) and consider their particular learning characteristics to make any necessary accommodations. Teachers should work in consultation with resource teachers, ESL/ELD teachers, and parents or guardians to determine appropriate accommodations as they work through the course in order to achieve the IEP expectations.

Accommodations for ESL/ELD Students

- Have ESL students work in pairs, with peer tutors, with classmates that have the same linguistic background, or with cooperative supportive groups, where they are more likely to improve their use of English. Brainstorm in groups using the students' first language if their usage of English is limited.
- Use peer conferencing to reinforce instructions or information.
- Provide reference notes, outlines of critical information, models of charts, timelines or diagrams.
- Use visuals to illustrate definitions for the students' dictionary of terms.
- Pair written instructions with verbal instructions. Provide visual or auditory cues.
- Simplify instructions. Highlight key words or phrases.
- Reinforce main ideas by using the think/pair/share peer-assessment strategy.
- Provide opportunities for students to practice oral presentation skills.
- Ask an ESL/ELD teacher to review questions, assignments, or assessment instruments.

Accommodations for Students with Learning Disabilities

- Provide extensive student-teacher conferencing.
- Provide a list of terms (possibly simplified) before an activity begins.
- Modify handouts in terms of the terminology and content used, as well as the size and typeface of the selected font. Allow plenty of space for written responses.

- Allow assignments to be completed in alternate formats or using longer timelines.
- Keep manipulatives, grid paper, formula sheets, and other aids available for needs that arise.
- Provide the students with oral pre-planning of activities.
- Pair students in order to provide appropriate support, for the identified student.
- Contact parents or guardians for support and suggestions.

Accommodations for Gifted Students

- Pose open-ended questions that require higher-level thinking.
- Accept ideas and suggestions from students and expand on them.
- Model creative thinking strategies, (e.g., decision-making and evaluation of problem-solving approaches).
- Create flexible instructional groups.
- Encourage independent investigations and projects.
- Facilitate original and independent problems and solutions.
- Take the time to explain the nature of errors.
- Find academic and community mentors for students.

Resources

This course profile has been provided as a resource to aid the teacher in delivering the curriculum. Through the discretionary use of other materials, the teacher can enrich, remediate, or otherwise supplement their students' education. The following is a partial list of widely available resources.

Software (Ministry-Licensed)

Geometer's Sketchpad (dynamic geometry)

Maple (word processor/programming)

Mastering Calculus (concept and skill development)

Math Trek (concept and skill development)

Virtual Tiles (algebraic concept and skill development)

Zap-a-Graph (graphing)

Internet sites

Note: The URLs for the websites have been verified by the writer prior to publication. Given the frequency with which these designations change, teachers should always verify the websites prior to assigning them for student use.

Canadian Education on the Web (http://www.oise.on.ca/~mpress/eduweb.html)
A compendium of Canadian education-related resources maintained by Marian Press at the Ontario Institute for Studies in Education/University of Toronto.

Education Network of Ontario (http://www.enoreo.on.ca/)
ENO is a computer communications network for everyone who works in elementary and secondary education in Ontario. Members have private accounts, which entitle them to participate in moderated newsgroups on education topics and training.

Hewlett-Packard (http://www.hp.com/calculators/)

National Council of Teachers of Mathematics (http://www.nctm.org)

Ontario Association of Mathematics Educators (http://www.oame.on.ca)

Ontario Curriculum Centre (http://www.curriculum.org)
A non-profit organization established to coordinate the sharing of teaching materials across Ontario.

Texas Instruments (http://www.ti.com/calc/docs)

Print

Burz, H.L., Marshall. K. *Performance-Based Curriculum for Mathematics*. California: Sage. 1996.

Concerning Assessment and Reflective Evaluation (CARE) Package (download from http://www.oame.on.ca)

MathMania: Adventures in Mathematics. London, ON: Gadanidis, G. ISSN 0843-851X.

The Mathematics Teacher. Reston, VA: National Council of Teachers of Mathematics (NCTM). ISSN 0025-5769

Connecting Mathematics: Addenda Series, Grades 9-12. NCTM Reston, VA: National Council of Teachers of Mathematics (NCTM), 1991. ISBN 0-87353-327-5

O.S.S.T.F. *Quality Assessment*. Toronto: Educational Services Committee. 1999.

Stiggins, R. *Classroom Assessment for Student Success*. Washington, DC: National Education Association of the United States. 1998.

Taggart, G. (Ed.) *Rubrics – A Handbook for Construction and Use*. Lancaster, PA: Techonomic Publishing. 1998.

OSS Considerations

The following list of resources will support many of the Ontario Secondary School Policies as well as the Ontario Catholic Secondary School Graduate Expectations.

- Ministry of Education Policy and Reference Documents

Choices into Action: Guidance and Career Education Program Policy

Cooperative Education: Policies and Procedures for Ontario Secondary Schools

Individual Education Plans: Standards for Development, Program Planning, and Implementation, 2000

Mathematics, Grades 9-10

Mathematics, Grades 11-12

Ontario Schools Code of Conduct

Ontario Secondary Schools, Grades 9-12: Program and Diploma Requirements

Program Planning and Assessment, Grades 9-12

Violence-Free Schools Policy

The Ministry of Education has also published several resource documents, brochures, and policy/program memoranda in support of its OSS policies. They are available online at the Ministry of Education website, http://www.edu.gov.on.ca/eng/document/document.html.

- Publications Concerning Faith Development

Blueprints (Catholic Curriculum Cooperative - Central Ontario Region)

Catholicity Across The Curriculum (Ontario Catholic School Trustees' Association)

Educating the Soul (Institute for Catholic Education)

Ontario Catholic Secondary School Graduate Expectations (Institute for Catholic Education)

This Moment of Promise (Ontario Conference of Catholic Bishops)

- Career Goals/Cooperative Education Programs

Ontario Youth Apprenticeship Program

Youth Employment Skills Program

- Community Partnerships

Refer to local board policies (e.g., *Relations with Business - Corporate Donations, Sponsorships and Agreements*).

Coded Expectations, Functions and Relations, Grade 11, University Preparation, MCR3U

Financial Applications of Sequences and Series

Overall Expectations

FAV.01 · solve problems involving arithmetic and geometric sequences and series;
FAV.02 · solve problems involving compound interest and annuities;
FAV.03 · solve problems involving financial decision making, using spreadsheets or other appropriate technology.

Specific Expectations

Solving Problems Involving Arithmetic and Geometric Sequences and Series

FA1.01 – write terms of a sequence, given the formula for the *n*th term or given a recursion formula;
FA1.02 – determine a formula for the *n*th term of a given sequence (e.g., the *n*th term of the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ is $\frac{n}{n+1}$);
FA1.03 – identify sequences as arithmetic or geometric, or neither;
FA1.04 – determine the value of any term in an arithmetic or a geometric sequence, using the formula for the *n*th term of the sequence;
FA1.05 – determine the sum of the terms of an arithmetic or a geometric series, using appropriate formulas and techniques.

Solving Problems Involving Compound Interest and Annuities

FA2.01 – derive the formulas for compound interest and present value, the amount of an ordinary annuity, and the present value of an ordinary annuity, using the formulas for the *n*th term of a geometric sequence and the sum of the first *n* terms of a geometric series;
FA2.02 – solve problems involving compound interest and present value;
FA2.03 – solve problems involving the amount and the present value of an ordinary annuity;
FA2.04 – demonstrate an understanding of the relationships between simple interest, arithmetic sequences, and linear growth;
FA2.05 – demonstrate an understanding of the relationships between compound interest, geometric sequences, and exponential growth.

Solving Problems Involving Financial Decision Making

FA3.01 – analyse the effects of changing the conditions in long-term savings plans (e.g., altering the frequency of deposits, the amount of deposit, the interest rate, the compounding period, or a combination of these) (*Sample problem*: Compare the results of making an annual deposit of $1000 to an RRSP, beginning at age 20, with the results of making an annual deposit of $3000, beginning at age 50);
FA3.02 – describe the manner in which interest is calculated on a mortgage (i.e., compounded semi-annually but calculated monthly) and compare this with the method of interest compounded monthly and calculated monthly;
FA3.03 – generate amortization tables for mortgages, using spreadsheets or other appropriate software;
FA3.04 – analyse the effects of changing the conditions of a mortgage (e.g., the effect on the length of time needed to pay off the mortgage of changing the payment frequency or the interest rate);
FA3.05 – communicate the solutions to problems and the findings of investigations with clarity and justification.

Trigonometric Functions

Overall Expectations

TFV.01 · solve problems involving the sine law and the cosine law in oblique triangles;
TFV.02 · demonstrate an understanding of the meaning and application of radian measure;
TFV.03 · determine, through investigation, the relationships between the graphs and the equations of sinusoidal functions;
TFV.04 · solve problems involving models of sinusoidal functions drawn from a variety of applications.

Specific Expectations

Solving Problems Involving the Sine Law and the Cosine Law in Oblique Triangles

TF1.01 – determine the sine, cosine, and tangent of angles greater than 90°, using a suitable technique (e.g., related angles, the unit circle), and determine two angles that correspond to a given single trigonometric function value;

TF1.02 – solve problems in two dimensions and three dimensions involving right triangles and oblique triangles, using the primary trigonometric ratios, the cosine law, and the sine law (including the ambiguous case).

Understanding the Meaning and Application of Radian Measure

TF2.01 – define the term *radian measure*;

TF2.02 – describe the relationship between radian measure and degree measure;

TF2.03 – represent, in applications, radian measure in exact form as an expression involving π (e.g., $\frac{\pi}{3}$, 2π) and in approximate form as a real number (e.g., 1.05);

TF2.04 – determine the exact values of the sine, cosine, and tangent of the special angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ and their multiples less than or equal to 2π;

TF2.05 – prove simple identities, using the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, and the quotient relation, $\tan x = \frac{\sin x}{\cos x}$;

TF2.06 – solve linear and quadratic trigonometric equations (e.g., $6\cos^2 x - \sin x - 4 = 0$) on the interval $0 \le x \le 2\pi$;

TF2.07 – demonstrate facility in the use of radian measure in solving equations and in graphing.

Investigating the Relationships Between the Graphs and the Equations of Sinusoidal Functions

TF3.01 – sketch the graphs of $y = \sin x$ and $y = \cos x$, and describe their periodic properties;

TF3.02 – determine, through investigation, using graphing calculators or graphing software, the effect of simple transformations (e.g., translations, reflections, stretches) on the graphs and equations of $y = \sin x$ and $y = \cos x$;

TF3.03 – determine the amplitude, period, phase shift, domain, and range of sinusoidal functions whose equations are given in the form $y = a\sin(kx + d) + c$ or $y = a\cos(kx + d) + c$;

TF3.04 – sketch the graphs of simple sinusoidal functions [e.g., $y = a\sin x$, $y = \cos kx$, $y = \sin(x + d)$, $y = a\cos kx + c$];

TF3.05 – write the equation of a sinusoidal function, given its graph and given its properties;

TF3.06 – sketch the graph of $y = \tan x$; identify the period, domain, and range of the function; and explain the occurrence of asymptotes.

Solving Problems Involving Models of Sinusoidal Functions

TF4.01 – determine, through investigation, the periodic properties of various models (e.g., the table of values, the graph, the equation) of sinusoidal functions drawn from a variety of applications;

TF4.02 – explain the relationship between the properties of a sinusoidal function and the parameters of its equation, within the context of an application, and over a restricted domain;

TF4.03 – predict the effects on the mathematical model of an application involving sinusoidal functions when the conditions in the application are varied;

TF4.04 – pose and solve problems related to models of sinusoidal functions drawn from a variety of applications, and communicate the solutions with clarity and justification, using appropriate mathematical forms.

Tools for Operating and Communicating with Functions

Overall Expectations

OCV.01 · demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;

OCV.02 · demonstrate an understanding of inverses and transformations of functions and facility in the use of function notation;

OCV.03 · communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations

Manipulating Polynomials, Rational Expressions, and Exponential Expressions

OC1.01 – solve first-degree inequalities and represent the solutions on number lines;

OC1.02 – add, subtract, and multiply polynomials;

OC1.03 – determine the maximum or minimum value of a quadratic function whose equation is given in the form $y = ax^2 + bx + c$, using the algebraic method of completing the square;

OC1.04 – identify the structure of the complex number system and express complex numbers in the form $a + bi$, where $i^2 = -1$ (e.g., $4i$, $3 - 2i$);

OC1.05 – determine the real or complex roots of quadratic equations, using an appropriate method (e.g., factoring, the quadratic formula, completing the square), and relate the roots to the *x*-intercepts of the graph of the corresponding function;

OC1.06 – add, subtract, multiply, and divide complex numbers in rectangular form;

OC1.07 – add, subtract, multiply, and divide rational expressions, and state the restrictions on the variable values;

OC1.08 – simplify and evaluate expressions containing integer and rational exponents, using the laws of exponents;

OC1.09 – solve exponential equations (e.g., $4^x = 8^{x+3}$, $2^{2x} - 2^x = 12$).

Understanding Inverses and Transformations and Using Function Notation

OC2.01 – define the term *function*;

OC2.02 – demonstrate facility in the use of function notation for substituting into and evaluating functions;

OC2.03 – determine, through investigation, the properties of the functions defined by $f(x) = \sqrt{x}$ [e.g., domain, range, relationship to $f(x) = x^2$] and $f(x) = \frac{1}{x}$ [e.g., domain, range, relationship to $f(x) = x$.];

OC2.04 – explain the relationship between a function and its inverse (i.e., symmetry of their graphs in the line $y = x$; the interchange of x and y in the equation of the function; the interchanges of the domain and range), using examples drawn from linear and quadratic functions, and from the functions $f(x) = \sqrt{x}$ and $f(x) = \frac{1}{x}$;

OC2.05 – represent inverse functions, using function notation, where appropriate;

OC2.06 – represent transformations (e.g., translations, reflections, stretches) of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$, using function notation;

OC2.07 – describe, by interpreting function notation, the relationship between the graph of a function and its image under one or more transformations;

OC2.08 – state the domain and range of transformations of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$.

Communicating Mathematical Reasoning

OC3.01 – explain mathematical processes, methods of solution, and concepts clearly to others;

OC3.02 – present problems and their solutions to a group, and answer questions about the problems and the solutions;

OC3.03 – communicate solutions to problems and to findings of investigations clearly and concisely, orally and in writing, using an effective integration of essay and mathematical forms;

OC3.04 – demonstrate the correct use of mathematical language, symbols, visuals (e.g., diagrams, graphs), and conventions;

OC3.05 – use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Investigations of Loci and Conics

Overall Expectations

LCV.01 · represent loci, using various models (e.g., a verbal description, a diagram, a dynamic model, an equation);

LCV.02 · determine the equation and the key features of a conic;

LCV.03 · solve problems involving applications of the conics.

Specific Expectations

Representing Loci

LC1.01 – construct a geometric model (e.g., a diagram created by hand, a diagram created by using dynamic geometry software) to represent a described locus of points; determine the properties of the geometric model; and use the properties to interpret the locus (e.g., the locus of points equidistant from two fixed points is the right bisector of the line segment joining the two fixed points);

LC1.02 – explain the process used in constructing a geometric model of a described locus;

LC1.03 – determine an equation to represent a described locus [e.g., determine the equation of the locus of points equidistant from (–2, 7) and (5, 4)];

LC1.04 – construct geometric models to represent the locus definitions of the conics;

LC1.05 – determine equations for conics from their locus definitions, by hand for simple particular cases [e.g., determine the equation of the locus of points the sum of whose distances from (–3, 0) and (3, 0) is 10].

Determining the Equation and the Key Features of a Conic

LC2.01 – identify the standard forms for the equations of parabolas, circles, ellipses, and hyperbolas having centres at (0, 0) and at (h, k);

LC2.02 – identify the type of conic, given its equation in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$;

LC2.03 – determine the key features (e.g., the centre or the vertex, the focus or foci, the asymptotes, the lengths of the axes) of a conic whose equation is given in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, by hand in simple cases (e.g., $x^2 + 9y^2 - 6x + 36y - 36 = 0$);

LC2.04 – sketch the graph of a conic whose equation is given in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$;

LC2.05 – illustrate the conics as intersections of planes with cones, using concrete materials or technology.

Solving Problems Involving Applications of the Conics

LC3.01 – describe the importance, within applications, of the focus of a parabola, an ellipse, or a hyperbola (e.g., all incoming rays parallel to the axis of a parabolic antenna are reflected through the focus; the planets move in elliptical orbits with the sun at one of the foci);

LC3.02 – pose and solve problems drawn from a variety of applications involving conics, and communicate the solutions with clarity and justification (*Sample problem*: A parabolic antenna is 320 m wide at a distance of 50 m above its vertex. Determine the distance above the vertex of the focus of the antenna);

LC3.03 – solve problems involving the intersections of lines and conics.

Ontario Catholic School Graduate Expectations

The graduate is expected to be:

A Discerning Believer Formed in the Catholic Faith Community who

CGE1a -illustrates a basic understanding of the **saving story** of our Christian faith;
CGE1b -participates in the **sacramental life** of the church and demonstrates an understanding of the centrality of the Eucharist to our Catholic story;
CGE1c -actively reflects on **God's Word** as communicated through the Hebrew and Christian scriptures;
CGE1d -develops attitudes and values founded on Catholic **social teaching** and acts to promote social responsibility, human solidarity and the common good;
CGE1e -speaks the **language of life**... "recognizing that life is an unearned gift and that a person entrusted with life does not own it but that one is called to protect and cherish it." (Witnesses to Faith)
CGE1f -seeks intimacy with God and celebrates **communion** with God, others and creation through prayer and worship;
CGE1g -understands that one's purpose or **call in life** comes from God and strives to discern and live out this call throughout life's journey;
CGE1h -respects the **faith traditions**, world religions and the life-journeys **of all people of good will**;
CGE1i -integrates faith with life;
CGE1j -recognizes that "sin, human weakness, conflict and forgiveness are part of the human journey" and that the cross, the ultimate sign of forgiveness is at the heart of **redemption**. (Witnesses to Faith)

An Effective Communicator who
CGE2a -listens actively and critically to understand and learn in light of gospel values;
CGE2b -reads, understands and uses written materials effectively;
CGE2c -presents information and ideas clearly and honestly and with sensitivity to others;
CGE2d -writes and speaks fluently one or both of Canada's official languages;
CGE2e -uses and integrates the Catholic faith tradition, in the critical analysis of the arts, media, technology and information systems to enhance the quality of life.

A Reflective and Creative Thinker who
CGE3a -recognizes there is more grace in our world than sin and that hope is essential in facing all challenges;
CGE3b -creates, adapts, evaluates new ideas in light of the common good;
CGE3c -thinks reflectively and creatively to evaluate situations and solve problems;
CGE3d -makes decisions in light of gospel values with an informed moral conscience;
CGE3e -adopts a holistic approach to life by integrating learning from various subject areas and experience;
CGE3f -examines, evaluates and applies knowledge of interdependent systems (physical, political, ethical, socio-economic and ecological) for the development of a just and compassionate society.

A Self-Directed, Responsible, Life Long Learner who

CGE4a -demonstrates a confident and positive sense of self and respect for the dignity and welfare of others;
CGE4b -demonstrates flexibility and adaptability;
CGE4c -takes initiative and demonstrates Christian leadership;
CGE4d -responds to, manages and constructively influences change in a discerning manner;
CGE4e -sets appropriate goals and priorities in school, work and personal life;
CGE4f -applies effective communication, decision-making, problem-solving, time and resource management skills;
CGE4g -examines and reflects on one's personal values, abilities and aspirations influencing life's choices and opportunities;
CGE4h -participates in leisure and fitness activities for a balanced and healthy lifestyle.

A Collaborative Contributor who

CGE5a -works effectively as an interdependent team member;
CGE5b -thinks critically about the meaning and purpose of work;
CGE5c -develops one's God-given potential and makes a meaningful contribution to society;
CGE5d -finds meaning, dignity, fulfillment and vocation in work which contributes to the common good;
CGE5e -respects the rights, responsibilities and contributions of self and others;
CGE5f -exercises Christian leadership in the achievement of individual and group goals;
CGE5g -achieves excellence, originality, and integrity in one's own work and supports these qualities in the work of others;
CGE5h -applies skills for employability, self-employment and entrepreneurship relative to Christian vocation.

A Caring Family Member who

CGE6a -relates to family members in a loving, compassionate and respectful manner;
CGE6b -recognizes human intimacy and sexuality as God given gifts, to be used as the creator intended;
CGE6c -values and honours the important role of the family in society;
CGE6d -values and nurtures opportunities for family prayer;
CGE6e -ministers to the family, school, parish, and wider community through service.

A Responsible Citizen who

CGE7a -acts morally and legally as a person formed in Catholic traditions;
CGE7b -accepts accountability for one's own actions;
CGE7c -seeks and grants forgiveness;
CGE7d -promotes the sacredness of life;
CGE7e -witnesses Catholic social teaching by promoting equality, democracy, and solidarity for a just, peaceful and compassionate society;
CGE7f -respects and affirms the diversity and interdependence of the world's peoples and cultures;
CGE7g -respects and understands the history, cultural heritage and pluralism of today's contemporary society;
CGE7h -exercises the rights and responsibilities of Canadian citizenship;
CGE7i -respects the environment and uses resources wisely;
CGE7j -contributes to the common good.

Unit 1: Exploring Functions: Connecting Algebra and Geometry

Time: 18 hours

Unit Description

Students investigate quadratic functions and related concepts from algebraic and geometric perspectives, in order to deepen their understanding and prepare them for further explorations of functions and relations. A winter recreation theme is loosely woven throughout selected activities in the unit, providing a contextual framework for students to solve problems, both with and without the use of graphing technology. Students solve first-degree inequalities and graph their solutions on number lines. Skills involving operations with polynomials and rational expressions are consolidated, and then extended to the complex number system which is introduced in this unit. Students apply the method of completing the square in order to solve maximum/minimum problems involving quadratic functions. Algebraic and graphical methods are used to determine the roots of quadratic equations. The exponent laws are applied to expressions, which have powers containing integer and rational exponents. Students discover the nature of exponential functions and solve exponential equations.

Activity 1: A Range of Possibilities

Time: 75 minutes

Description

Students add, subtract, and multiply polynomials in inequalities. Students investigate solutions to inequalities using graphing calculator technology, and subsequently graph solutions on a number line.

Strand(s): Tools for Operating and Communicating with Functions

Overall Expectations

OCV.01 - demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;
OCV.03 - communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations

OC1.01 - solve first-degree inequalities and represent the solutions on number lines;
OC1.02 - add, subtract, and multiply polynomials;
OC3.04 - demonstrate the correct use of mathematical language, symbols, visuals (e.g., diagrams, graphs), and conventions;
OC3.05 - use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Prior Knowledge & Skills

- Solve first degree equations in one variable;
- Manipulate algebraic expressions;
- Perform basic graphing functions on a graphing calculator, including zooming and tracing.

Planning Notes

- Prepare worksheets.
- Students require graphing calculators.
- Teacher requires a graphing calculator with an overhead projection unit.

Teaching/Learning Strategies

Student Activity

Students:

- manipulate polynomials by adding, subtracting, and multiplying;
- use graphing calculators to visually investigate solutions of inequalities;
- solve inequalities algebraically and graph solutions on number lines.

Teacher Facilitation

- Begin with a discussion about the differences in solutions between equalities and inequalities. Using a graphing calculator with an overhead projection unit attached, project the graph of:
  a) $y = 2x - 6$. Direct the students to use the graph of this function to determine the root of the equation $2x - 6 = 0$ (use the zoom and trace features to locate the x-intercept).
  b) $2x - 6 > 0$. Discuss, and direct students to use the graph to solve the inequality. Use a test point to help determine the region defined by the inequality. Students should discover that x-values for which the graph is above the x-axis represent the solution set.
  c) $2x - 6 \# 0$. Repeat the process from (b). Students should discover that x-values for which the graph is on, or below, the x-axis represent the solution set.
- As a class, represent the solutions to (a), (b) and (c) on the real number line. The concept of *projecting the line onto the x-axis* may help students to visualize the connection between the Cartesian illustration, in two dimensions, and the corresponding one-dimensional graph on the real number line. Discuss differences and similarities in the solutions of (a), (b), and (c).
- Organize the class into small groups. Have them repeat the above investigation using the following equations: (a) $-4(x + 2) - 3(x + 4) = 3x$, (b) $-4(x + 2) - 3(x + 4) < 3x$, and (c) $-4(x + 2) - 3(x + 4) \$ 3x$. There are different ways to solve these. Encourage students to experiment with the calculator and devise their own technique.
- One approach is to graph two functions: $y_1 = -4(x + 2) - 3(x + 4)$ and $y_2 = 3x$. For clarity, use different line styles for each. The x-coordinate of the intersection point gives the solution to (a), while the set of x-values where y_1 appears below y_2 gives the solution to (b), and vice versa for (c), including the intersection point.
- Another approach is to move all non-zero terms to one side of the inequality, graph the corresponding linear relation, and focus on the x-intercept, as in the first example.
- Have students share their techniques with the class. After this, the teacher can model the algebraic techniques in order to verify the results, and provide an alternate solution approach.
- Distribute Sample Worksheet 1. Complete question 1 (parts i to iii) for the first equation on the worksheet, $3 - x < 6 - 2x$.
- Using graphing calculator technology, have students work in pairs to complete the rest of the worksheet, and additional worksheets, or appropriate exercises from the textbook.

Sample Worksheet 1

1. (a) $3 - x < 6 - 2x$
   (b) $3(2x - 1) - 2(x + 1) \# 3x + 8$
   (c) $3(4 - x) - 2 > 2(x - 3) + 6$
   i) Solve each inequality algebraically, ii) Represent the solution on a number line,
   iii) Confirm your solution set by graphing the inequality using a graphing calculator.
2. Explain how the solution sets of the following inequalities differ: (a) $x > 2$ (b) $x \$ 2$.
3. How does the nature of the solution set of the inequality $x \# -3$ differ, when represented on: (a) the real number line; (b) the Cartesian plane.
4. Consider the different techniques to solve inequalities. Discuss advantages and disadvantages of each.

Assessment & Evaluation of Student Achievement

Question 1, parts (i), (ii), and (iii) on Sample Worksheet 1 can be assessed for correct use of mathematical symbols, visuals, and conventions. The remaining questions can be assessed for Communication. Assessment should be of a formative nature in this activity.

Extension

Referring back to question 3, from Sample Worksheet 1, how would the nature of the solution sets in parts (a) and (b) differ from the solution set of x # -3, when represented on a 3-dimensional, x-y-z coordinate system, where z is an axis passing through the origin of the Cartesian plane, perpendicular to both the x and y axes?

Key Points to Look for in Answers to 3 and Extension: On the number line, x # -3 represents a ray of points to the left of $x = -3$. On the Cartesian Plane, x # -3 represents a 2-dimensional region of points or the area to the left of the line $x = -3$. In 3-space, x # -3 represents a 3-dimensional region of points on the space to the left of the plane $x = 3$ (which is parallel to the y-z plane).

Activity 2: Ski-Jumping to the Max!

Time: 150 minutes

Description

Students investigate the graphs of quadratic functions and examine their vertices and x-intercepts. Using graphing technology, students determine quadratic functions, which have desired maximum values. The method of completing the square is used to connect the algebraic and geometric significance of the vertex.

Strand(s): Tools for Operating and Communicating with Functions

Overall Expectations

OCV.01 - demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;
OCV.03 - communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations

OC1.03 - determine the maximum or minimum value of a quadratic function whose equation is given in the form $y = ax^2 + bx + c$, using the algebraic method of completing the square;
OC1.05 - determine the real or complex roots of quadratic equations, using an appropriate method (e.g., factoring, the quadratic formula, completing the square), and relate the roots to the x-intercepts of the graph of the corresponding function;
OC3.01 - explain mathematical processes, methods of solution, and concepts clearly to others;
OC3.05 - use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Prior Knowledge & Skills

- Complete the square in situations without fractions;
- Identify the vertex of a parabola expressed in the form $y = a(x - h)^2 + k$;
- Understand that the real roots of a quadratic equation are the x-intercepts of the graph of the corresponding quadratic function;
- Locate vertices and intercepts using graphing technology;
- Determine roots of quadratic equations, using the quadratic formula.

Planning Notes

- Prepare worksheets.
- Students require graphing calculators, or access to appropriate graphing software.
- Organize the class into pairs, or small, heterogeneous groups of about 3 or 4.

Teaching/Learning Strategies

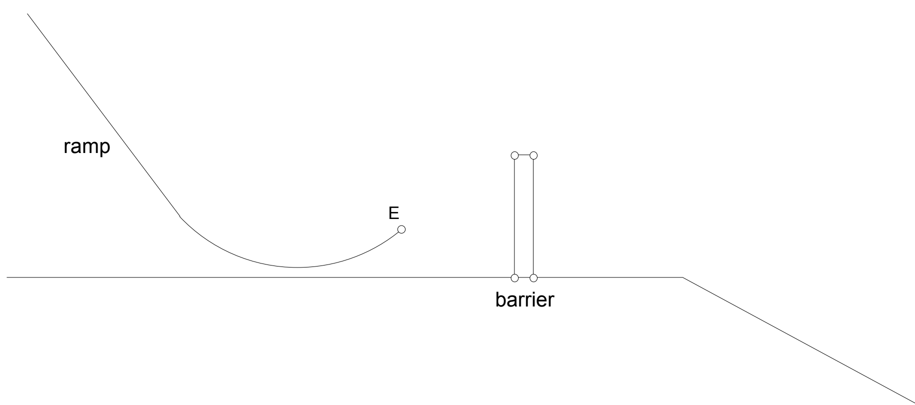
Student Activity

Students:

- use graphing technology to graph a parabola;
- identify the vertex of a parabola from the its graph;
- complete the square;
- identify characteristics of a parabola expressed in the form $y = a(x - h)^2 + k$.

Teacher Facilitation

- Introduce the following scenario to the class: A new ski-jumping stunt is planned, as shown below An expert ski jumper will accelerate down the ramp, and leave it at the “exit point”, *E*, which is 8 m above the horizontal ground. She will leave the ramp at an angle of inclination of 30° to the ground, with an initial speed of *v*. Ignoring air resistance, the height, *y*, of the ski jumper can be described by the equation $y = -4.9t^2 + 0.5vt + 8$, where *t* is the time, in seconds, starting from the time when the jumper leaves the ramp. The derivation of this equation is left as an exercise at the end of the activity. It should be discussed why “*t*” is a more appropriate variable than “*x*”, for this problem. The skier, an expert at such stunts, can control her exit speed, *v*, by adjusting her ski position and body tuck. She wants to determine the minimum speed she needs in order to clear the barrier with just a 1 metre margin of error, for dramatic effect.



- The barrier can be shifted closer to, or further from the ramp, however the barrier is not placed until after the optimum speed has been determined. Pose the question: “Why is this important?” Students should determine that as speed changes, the horizontal position of the maximum height will shift.
- Starting with a barrier height of 15 m, students are to use a graphing calculator to determine the optimum speed that will allow for a successful jump. Varying the parameter *v* through systematic trials, investigating the resulting parabolas and using the trace function to locate maxima is a typical approach to solving the problem.
- After students have discovered the optimum speed, reassemble the class. Have one or two groups present their method to the class. It should be determined that, for a barrier height of 15 m, a speed of approximately 25.1 m/s is required, which corresponds to the function $y = -4.9t^2 + 12.55t + 8$.
- At this point, review the algorithm of completing the square and apply it to one or two simple examples, which do not involve fractions or decimals.

- Now guide students through the process of completing the square on the function $y = -4.9t^2 + 12.55t + 8$, where $a = -4.9$, $b = 12.55$ and $c = 8$. Discuss the validity of rounding in this particular situation, with some attention to the carrying of an appropriate, or reasonable, number of significant digits.

  $y = -4.9(t^2 + 2.56t) + 8$ *factor out "a" from the first two terms*

  $y = -4.9(t^2 + 2.56t + 1.28^2 - 1.28^2) + 8$ *add and subtract* $(\frac{b}{2a})^2$ *inside the brackets*

  continue completing the square…

  $\mathbf{y = -4.9(t - 1.3)^2 + 16}$
- Prompt the students to look at the terms in this equation. Ask the following: "Can you see any possible relationships between these terms and the graph of the function? What is the physical significance of the values 1.3 and 16?" Have students repeat the exercise for a new barrier height. They are to determine the optimum speed, and then complete the square, in order to express the function in the form shown above. Assign a different barrier height (e.g., 10, 12, 18, 20, etc.) to each group, in order to generalize the results more readily. Once groups have had a chance to do this, have the class share with each other, using Sample Worksheet 1. Then have students complete the questions, alone or in small groups. The teacher may need to review, with one or two examples, application of the quadratic formula, in order to assist with question 3.

Sample Worksheet 1

Barrier height (m)	**Optimum Initial Speed** (m/s)	**Quadratic Equation Standard Form**	**Quadratic Equation** $y = a(x - h)^2 + k$
10			
12			
15	25.1	$y = -4.9t^2 + 12.55t + 8$	$y = -4.9(t - 1.3)^2 + 16$
18			
20			

1. What is the physical significance of h and k, in the equation $y = -4.9(t - h)^2 + k$?
2. For $y = -4.9t^2 + 0.5vt + 8$, what is the physical significance of the 8? Prove this is true.
3. Determine the t-intercepts of the quadratic equation $-4.9t^2 + 12.55t + 8 = 0$, by using the quadratic formula, and by using a graphing calculator.
4. The term "hang time" refers to the amount of time a ski jumper is in the air. Considering the diagram provided, how is the right t-intercept on the graph related to the hang time?
5. What is the physical significance of the y-intercept of the graph?
6. What is the geometric significance of the left t-intercept of the graph? Why does this point have no real physical meaning?
7. Why is it important that the barrier be moveable, closer to, or further from the ramp, when changing the barrier's height?

Key Points to Look for in Answers to Questions

1. h is the time at which the jumper reaches the maximum height; k is the maximum height.
2. 8 is the initial height of the jumper, at time zero. One way to prove this is to set $t = 0$ in the equation and simplify for y.
3. The roots are –0.53 seconds and 3.1 seconds.
4. If the ground were horizontal between the ramp and the landing point then this would represent hang time. Typically the skier will land, for safety purposes, where the land is sloping downwards, so the hang time will be greater than this value.
5. This point represents the initial height of the jumper, and corresponds to $t = 0$.

6. This point represents the other point at which the jumper would be at ground level, if the parabolic function were extended to the left of the y-axis. Because this corresponds to $t < 0$, there is no physical meaning for this point. The skier's motion is not accurately described by this function prior to $t = 0$.
7. The skier will reach maximum height at different horizontal distances from the ramp, as the initial speed changes.

Assessment & Evaluation of Student Achievement

Depending on the format of assessing the various components of the activity (e.g., brief oral presentations, written submissions, etc.), the teacher may use all, or parts of the following rubric. A category addressing Knowledge has been included in the rubric, however the teacher may wish to replace or augment the Knowledge assessment with a quiz, focussing on the process of completing the square. It is recommended that the assessment in this activity be used formatively, with the option of using a similar rubric for a summative performance task later. Refer also to the "Generic Rubrics" for Communication and Thinking/Inquiry/Problem Solving, from the OAME/OMCA "CARE Package", which can be downloaded from http://www.oame.on.ca.

Criteria	Level 1 (50-59%)	Level 2 (60-69%)	Level 3 (70-79%)	Level 4 (80-100%)
Inquiry Determines optimum speeds and particular quadratic functions	- uses an approach which demonstrates a limited understanding of the problem	- uses an approach which demonstrates some understanding of the problem	- uses an approach which demonstrates a considerable understanding of the problem	- uses an approach which demonstrates a thorough understanding of the problem
Application Uses graphing technology effectively, (e.g., sets display window, zoom, trace, etc.)	- uses graphing technology with limited effectiveness	- uses graphing technology with some effectiveness	- uses graphing technology with considerable effectiveness	- uses graphing technology with a high degree of effectiveness
Knowledge Understands how to complete the square to optimize a quadratic function	- limited understanding of how to complete the square to optimize a quadratic function	- some understanding of how to complete the square to optimize a quadratic function	- considerable understanding of how to complete the square to optimize a quadratic function	- thorough understanding of how to complete the square to optimize a quadratic function
Communication Clarity of explanations	- demonstrates limited clarity in making explanations	- demonstrates some clarity in making explanations	- demonstrates considerable clarity in making explanations	- demonstrates thorough clarity in making explanations

Extensions

1. Students could investigate the effects of changing the ramp's angle of inclination. The term $0.5v$ in the original equation comes from the vertical component of the velocity, $\mathbf{v}$, when the jumper leaves the ramp: $v = *\mathbf{v}* \sin 30°$. By varying this angle, students could analyse the effect this has on maximum height and hang time.

2. Have students derive the formula $y = -4.9t^2 + 0.5vt + 8$ from the physics formula $y = 0.5at^2 + v_y t + d$, where a is the acceleration due to gravity, -9.8 m/s^2, v_y is the vertical component of the velocity (derived above) and d is the initial height of the skier.

Activity 3: Rooting Around the Parabola

Time: 150 minutes

Description

Students investigate quadratic functions and examine their *x*-intercepts. Using graphing technology, conceptual connections are drawn between the *x*-intercepts of a quadratic function, and the real or complex roots of its corresponding quadratic equation.

Strand(s): Tools for Operating and Communicating with Functions

Overall Expectations

OCV.01 - demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;

OCV.03 - communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations

OC1.04 - identify the structure of the complex number system and express complex numbers in the form $a + bi$, where $i^2 = -1$ (e.g., $4i$, $3 - 2i$);

OC1.05 - determine the real or complex roots of quadratic equations, using an appropriate method (e.g., factoring, the quadratic formula, completing the square), and relate the roots to the *x*-intercepts of the graph of the corresponding function;

OC3.02 - present problems and their solutions to a group, and answer questions about the problems and the solutions;

OC3.05 - use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Prior Knowledge & Skills

- Complete the square on quadratic expressions with no fractions or decimals;
- Identify the vertex of a parabola expressed in the form $y = a(x - h)^2 + k$;
- Understand that the real roots of a quadratic equation are the *x*-intercepts of the graph of the corresponding quadratic function;
- Locate vertices and intercepts using graphing technology;
- Use the quadratic formula to solve for real roots of a quadratic equation.

Planning Notes

- Students require graphing calculators, or appropriate graphing software on a computer.
- Appropriate overhead projection technology will be required.
- Organize the class into pairs, or small, heterogeneous groups of about three or four.

Teaching/Learning Strategies

Student Activity

Students:

- use graphing technology to graph parabolas;
- determine the roots of a parabola by visually identifying the *x*-intercepts of its graph;
- algebraically determine the roots of a quadratic equation using an appropriate method;

- examine complex roots of a quadratic equation;
- conjecture and test an hypothesis;
- design and carry out an investigation using graphing technology.

Teacher Facilitation

- Review with the class the connection between real roots of quadratic equations and the x-intercepts of their corresponding quadratic functions, using simple examples.
- Demonstrate that the value of the discriminant, $b^2 - 4ac$, in the quadratic formula gives information regarding the nature of the roots of a quadratic equation, with a few simple examples. Draw connections between the information provided by the discriminant to the x-intercepts of the corresponding functions. Pose the question, "How do we interpret roots of quadratic equations whose corresponding functions do not intersect the x-axis?"
- To set the stage for the investigation, work with a simple quadratic function, such as $y_1 = (x - 3)^2 + 4$. Show that the graph clearly does not intersect the x-axis.
- Next, expand the corresponding quadratic equation to produce $x^2 - 6x + 13 = 0$ and apply the quadratic formula to introduce the concept of complex roots, in this case: $3 \pm 2i$. Take it to the point where you get $x = \frac{6 \pm \sqrt{-16}}{2}$, and then interrupt the solution with the following.
- Engage in an introductory discussion of the complex number system. Explain that any complex number can be expressed in the form $a + bi$, where $\underline{a}$ is the real part, b is the imaginary part, and i is defined implicitly by $i^2 = -1$. Provide a mathematical context for the discussion by identifying complex numbers as a set of numbers of which the real, rational, irrational, integer, whole, and natural number sets are subsets. A broader context can be drawn by identifying specific areas of study in engineering, physical sciences, and mathematics, in which complex numbers are used to model algebraic and geometric concepts. Some examples include: alternating current and voltage in electrical power systems, quantum mechanics (study of the nature of matter at the sub-atomic level), electro-optics, and fractal models such as the Mandelbrot Set. The teacher may wish to consolidate the basic concepts using one or two additional examples at this point.
- Return attention to the function $y_1 = (x - 3)^2 + 4$, and finish deriving the complex roots of the corresponding quadratic equation: $3 \pm 2i$.
- Draw a comparison to the graph of $y_2 = -(x - 3)^2 + 4$, which is a reflection of the graph of the function y_1, in the line $y = 4$.
- Engage the students in a class discussion including the following: "How are these two functions related to each other, algebraically and geometrically? What are the x-intercepts of y_2?" As a class, algebraically determine the roots of the corresponding quadratic equation $(x - 3)^2 + 4 = 0$.
- The following part of the activity should be performed within small groups. Have the students do the following: Given two functions of the form $y_1 = a(x - h)^2 + k$, and $y_2 = -a(x - h)^2 + \text{k}$, conjecture a relationship between the roots of their corresponding quadratic equations. Design and carry out an investigation, which will serve to test your hypothesis, using appropriate graphing technology. Analyse a number of different quadratic functions in all four quadrants. **Note:** Sample hypotheses/conclusions are provided, at each achievement level, at the end of the Assessment & Evaluation of Student Achievement section below.
- Once groups have had time to perform the investigation, have some groups present their findings to the class. Their presentation should include their stated hypothesis, the approach to the investigation, with a demonstration using projected graphing technology, and their findings. Classmates should be encouraged to ask questions during the presentation and presenters should defend their arguments and answer questions from the class.
- Have students answer the questions below, and assign appropriate skill building exercises.

Follow Up Questions

1. Consider the following statement: "If the discriminant of a quadratic equation is less than zero, then the quadratic equation has no roots." Discuss the accuracy of this statement, based on your knowledge of various number sets.
2. Given a function in the form $y = a(x - h)^2 + k$, how can you easily determine the nature of the roots of the corresponding quadratic equation, by mentally analysing the terms?

Key Points to Look for in Answers to Questions

1. True statement if you confine the analysis to real numbers. There are complex roots.
2. If $ak > 0$ (a and k have the same sign), there are complex roots. If $ak < 0$, there are real roots.

Assessment & Evaluation of Student Achievement

The teacher may use all or parts of the following rubric for assessment which should generally be formative. Refer also to the "Generic Rubrics" for Communication and Thinking/Inquiry/Problem Solving, from the OAME/OMCA "CARE Package," which can be downloaded from http://www.oame.on.ca.

Criteria	Level 1 (50-59%)	Level 2 (60-69%)	Level 3 (70-79%)	Level 4 (80-100%)
Inquiry* Conjectures an hypothesis related to the investigation	- statement reveals limited understanding of the purpose of the investigation	- statement reveals some understanding of the purpose of the investigation	- statement reveals a considerable understanding of the purpose of the investigation	- statement reveals thorough understanding of the purpose of the investigation
Inquiry Performs an investigation	- investigates with limited effectiveness	- investigates with some effectiveness	- investigates with considerable effectiveness	- investigates with a high degree of effectiveness and poses extending questions
Inquiry* Arrives at a conclusion	- demonstrates limited ability to arrive at a conclusion	- demonstrates some ability to arrive at a conclusion	- demonstrates considerable ability to arrive at a conclusion	- demonstrates a high degree of ability to arrive at a conclusion
Application Uses graphing technology effectively, (e.g., sets display window, zoom, trace, etc.)	- uses graphing technology with limited effectiveness	- uses graphing technology with some effectiveness	- uses graphing technology with considerable effectiveness	- uses graphing technology with a high degree of effectiveness
Communication Clearly explains answers to questions (oral and written) related to the activity	- explains with limited clarity	- explains with some clarity	- explains with considerable clarity	- explains with a high degree of clarity

*Sample Hypotheses/Conclusions

Note: A student whose achievement is below level 1 (50%) has not met the expectations for this assignment or activity.

Level 1: The quadratic equation corresponding to one of the two functions will have real roots. The other one will have complex roots.
Level 2: The quadratic equation corresponding to the function whose graph intersects the x-axis will have real roots. The other will have complex roots.
Level 3: The quadratic equation corresponding to the function whose graph intersects the x-axis twice will have real roots. The other will have complex roots. In the case where $k = 0$, both equations will have one real root, and this root is the same for both equations.
Level 4: In the case where $k = 0$, both equations will have one real root, and this root is the same for both equations. If one quadratic equation has complex roots $a \pm bi$, then the other will have real roots r_1 and r_2, where $|a - r_1| = |a - r_2| = b$. Students who can otherwise, in less sophisticated terms, correctly quantify the relationship between the real and complex roots of the two quadratic equations, deserve Level 4 credit.

Extensions

1. A similar investigation can be carried out for other types of functions, (e.g., absolute value, square root).
2. (a) Show that for the function $y = a(x - h)^2 + k$, that the roots are given by the equation:

   $$x = \frac{2ah \pm \sqrt{-4ak}}{2a}$$

   (optional hint: expand the right hand side and substitute into the Quadratic Formula)
2. (b) Show that the result in 2(a) yields real roots when $ak < 0$ and complex roots when $ak > 0$ (*optional hint: focus on the radicand and consider cases*)
3. Introduce the complex plane as a coordinate system for graphing complex numbers.

Activity 4: Complex Basics are Basically not Complex!

Time: 150 minutes

Description

Students investigate complex numbers graphically on the complex plane using graphing calculators. Students apply the properties of complex numbers by adding, subtracting, multiplying and dividing complex numbers in the form $a + bi$, and express results in simplest form.

Strand(s) & Learning Expectations

Strand(s): Tools for Operating and Communicating with Functions

Overall Expectations

OCV.01 - demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;
OCV.03 - communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations

OC1.06 - add, subtract, multiply, and divide complex numbers in rectangular form;
OC3.04 - demonstrate the correct use of mathematical language, symbols, visuals (e.g., diagrams, graphs), and conventions;
OC3.05 - use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Prior Knowledge & Skills

- Understand basic graphing calculator functions;
- Simplify expressions with exponents.

Planning Notes

- Students may require graphing calculators.
- A graphing calculator, a graphing calculator projection unit, and an overhead projector may be used for the lesson demonstration involving the Complex plane.
- Prepare worksheets.

Teaching/Learning Strategies

Student Activity

Students:

- graph complex numbers on the Complex plane;
- add, subtract, multiply, and divide complex numbers in the form $a + bi$, and use i^2 and = -1 and $i = \sqrt{-1}$ to represent the results in simplest terms.

Teacher Facilitation

- The graphing parts of this activity can be done with or without graphing calculators. Instructions are provided to illustrate how they can be used to graph complex numbers.
- Begin with a general discussion about the applications of complex numbers. For example, complex numbers are used by mathematicians, physicists, and engineers, and have applications in current circuits, in quantum mechanics, and in aerodynamic design. Refer to the Teacher Facilitation section of Activity 3, for additional, specific examples of where complex numbers are used.
- Introduce the properties $i^2 = -1$, and $i = \sqrt{-1}$, and the complex number form $a + bi$.
- Explain how a complex number, such as $2 + 3i$, can be represented by a point on the complex plane. Using a graphing calculator and a graphing calculator projection unit, with an overhead projector, provide visual representation of complex numbers on the complex plane. Do this as follows: To plot the complex number $a + bi$, use a to replace x, b to replace y, and use the normal point plotting function. Now the rectangular grid can be thought of as the complex plane, where the x- and y-axes are replaced by the real and imaginary axes, respectively. Then discuss the similarities and differences between the Cartesian and Complex planes.
- Explain the concept of conjugates. Graphically represent a complex number and its conjugate $\left[e.g., (8-3i) \text{ and } (8+3i)\right]$ on the complex plane and discuss the visual representation. Briefly explore the properties of conjugates (i.e., symmetry in the Real axis).
- As a class, complete an example adding two complex numbers $\left[e.g., (2+3i)+(4-6i)\right]$. Link student's previous knowledge of collecting like terms to simplifying complex numbers $a + bi$, by collecting real and imaginary terms. With the teacher at the overhead projection unit and the students on their own graphing calculator, graph the two original numbers and the result on the graphing calculator. Discuss the visual representation. Repeat this process with examples of subtracting two complex numbers $\left[e.g., (4-2i)-(9+5i)\right]$, multiplying two complex numbers $\left[e.g., (7+i)(8-4i)\right]$, and dividing two complex numbers $\left[e.g., \left(\frac{6-8i}{2i}\right) \text{and} \left(\frac{5-3i}{4+i}\right)\right]$.
- Have students work in pairs to complete Sample Worksheet 1.
- Distribute additional worksheets, or assign appropriate work from the textbook.

Sample Worksheet 1

1. a) Are any numbers that are in the complex number system also in the real number system?
   b) Are any numbers that are in the real number system also in the complex number system?
   c) Justify your answers with examples.

2. a) Investigate the pattern which emerges by looking at successive powers of i^n, for natural n. Discuss both the algebraic results and their graphic representations.
   b) Use the results to simplify each of the following: i^{26}, i^{12}, i^{27}, and i^{13}.
   c) Summarize your results in (a) and (b) by explaining the pattern that emerges with odd and even exponents.
   d) Predict the value of i^{87} , then confirm your result through calculation.
3. The conjugate of $(2 + 5i)$ is $(2 – 5i)$.
   a) Calculate the product of these conjugates.
   b) What type of number is your result? (i.e., a real number? a complex number?)
   c) Explain why this type of result will be the same when any pair of complex number conjugates are multiplied together.
4. Plot the following numbers on the Complex plane:
   a) $5i$ b) 6 c) $4 + 3i$ d) $-2 – 7i$
5. Perform the following complex number operations. Plot the original numbers and the results on the complex plane: a) $(2 – 9i) +(-3 + 6i)$ b) $(4 – 2i)(-3 + 8i)$ c) $\left(\frac{7-2i}{4+i}\right)$
6. When two complex numbers in the form $a + bi$ are multiplied together, is the result always a complex number? Explain your answer using examples (i.e., Can you create examples that show the result is: (i) a complex number? (ii) a real number?).
7. A quadratic equation has roots $1 \pm 2i$. Determine the equation of a corresponding quadratic function.
8. Find two numbers whose sum is 8 and product is 25.

Teacher Facilitation
Once students have had an opportunity to complete the worksheet, introduce the "*i*" key on the graphing calculator, and have students use this feature to check their answers. This is a good opportunity to introduce the "decimal to fraction" and "fraction to decimal" functions, as well.

Assessment & Evaluation of Student Achievement
Students hand in completed Sample Worksheet 1 to be assessed. Communication can be assessed in questions 1, and 3. Inquiry and Communication can be assessed in questions 2, 6, 7, and 8. Parts of the rubric from Activity 3 can be adapted for this activity. Knowledge/Understanding can be assessed throughout the worksheet, using an objective marking scheme. The assessment in this activity should be largely formative.

Extension
Students can perform an investigation comparing the following: illustration of operations (addition, subtraction, multiplication, and division) of real numbers on the real number line, and illustration of the same operations of complex numbers on the complex plane.

Activity 5: Can We Please be Rational?!
Time: 150 minutes

Description
A physical geometric model is used to introduce the concept of a rational expression. Students use scientific and graphing calculators, in order to investigate properties of rational functions. Students simplify, add, subtract, multiply, and divide rational expressions, and state restrictions on variables.

Strand(s): Tools for Operating and Communicating with Functions

Overall Expectations

OCV.01 - demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;

OCV.03 - communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations

OC1.07 - add, subtract, multiply, and divide rational expressions, and state the restrictions on the variable values;

OC3.01 - explain mathematical processes, methods of solution, and concepts clearly to others;

OC3.03 - communicate solutions to problems and to findings of investigations clearly and concisely, orally and in writing, using an effective integration of essay and mathematical forms;

OC3.04 - demonstrate the correct use of mathematical language, symbols, visuals, (e.g., diagrams, graphs, and conventions);

OC3.05 - use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Prior Knowledge & Skills

- Perform operations (i.e., adding, subtracting, multiplying, and dividing) with polynomials;
- Apply factoring techniques (including common factoring, difference of squares, trinomials in the form $ax^2 + bx + c$ where $a = 1$, and where $a \neq 1$).
- Use the zoom and trace features of a graphing calculator to determine points on a graph.

Planning Notes

- Students require graphing calculators.
- Prepare worksheets.

Teaching/Learning Strategies

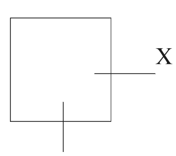
Student Activity

Students:

- simplify, add, subtract, multiply and divide rational expressions;
- determine the restrictions on the variables.

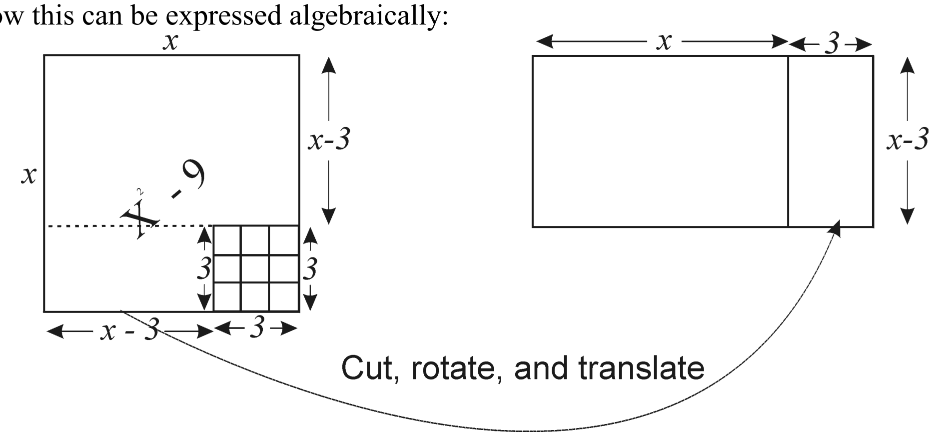
Teacher Facilitation

- Have the students think about the land on which some parking lots are located. A survey of their thoughts may result in the land identified as being in either a square or rectangular shape.
- Have them draw a square and then label a side length with x.
  Develop an expression for the area in terms of x.
  $A = x(x)$
  $A = x^2$



- Pose the question: "Suppose zoning laws made us reduce the area of the lot by 9, while one side was increased by 3. What would be the new width, in terms of the original length, x?"

- Show how this can be expressed algebraically:



- Discuss with the class the nature and properties of a rational expression.
- Have students perform the following numerical analysis:
  a) Use a scientific or graphing calculator to divide any given number by successively smaller numbers (getting closer and closer to zero), and describe what happens to the result. Explain why the calculator gives an error message when you try to divide by zero.
  b) Apply this concept to explain why the function $w = \frac{x^2 - 9}{x + 3}$ is said to be undefined for x = –3.
  c) Give a geometric explanation of the restriction on *x*.
- Refocus attention to the context of a physical lot once again.
- Pose the question: "Would an initial length of –2 be an acceptable value for this situation? Explain. Consider the mathematical model and the physical situation which it describes."
- Turn attention away from the physical model and consider another rational function: $y = \frac{1}{x - 3}$
- Have students perform the following graphical analysis:
  a) At standard zoom setting, graph the function.
  b) Noting an unusual behaviour around *x* = –3, zoom into this region, using the following window settings. In each case, trace the curve from both the left and right sides of *x* = 3. Note the *y*-coordinate as *x* gets very close to 3. Note also what happens to the *y*-coordinate if you are able to trace so that *x* = 3 exactly. Have students organize their findings in a chart:

Zoom Trial	X_{min}	X_{max}	Y_{min}	Y_{max}	**Observations**
1	2.5	3.5	–50	50	
2	2.9	3.1	–100	100	
3	2.99	3.01	–1,000	1,000	
4	2.999	3.001	–10,000	10,000	

- Discuss with students why the function is said to be undefined at *x* = 3. By introducing the terminology and concepts associated with asymptotes, you can lay a foundation for later studies in the Investigation of Loci and Conics unit. As a nice lead-in to future studies in Calculus, you can touch on the concept of the right and left-hand limits as *x* approaches 3.
- Return attention to the function $w = \frac{x^2 - 9}{x + 3}$ and have students graph this function, using the standard zoom setting. Remind students to toggle the other function off.

- Pose the following questions:
  a) "What shape is this graph?"
  b) "Why is it a line? Does the equation look linear?"
  c) "Does this graph make sense for $x = -3$, based on the earlier investigation?"
- At this point introduce the method of simplifying this rational expression, by factoring the numerator, and then dividing out the common factor, $x + 3$.
- Pose the question: "Does the equation look linear, now?"
- Discuss the need for recognizing a "hole" at the point $(-3, -6)$, in order to make the graph accurate. This is a good opportunity to identify the limitations of graphing technology and the need for critical thinking on the part of the user.
- Lead students through the (a) parts of questions 2 to 5 of Sample Worksheet 1. The teacher may need to review some factoring techniques with the class at this time.
- Assign the rest of the worksheet to be completed independently.
- Assign appropriate exercises in order to consolidate skills.

Sample Worksheet 1

1. Explain in your notes or journal the difference between mathematical restrictions on variables, and restrictions resulting from the physical nature of the area problem posed in class.
2. Consider the following expressions: a) $\frac{x+4}{x-7}$ b) $\frac{x+6}{x}$

   State the restrictions on each expression. If the denominators were representing a length of an object, would your answers be mathematical restrictions, physical ones, or both?
3. Another type of algebraic situation involves two or more variables in the denominator. State the restrictions on the following: a) $\frac{x+3y}{xy}$ b) $\frac{x+3y}{x-5y}$

   (**hint:** What makes the denominator equal zero?)
4. For each of the following, factor the numerator and denominator, state the restrictions, and then simplify (that is divide out common factors from the numerator and denominator).

   a) $\frac{2x+4}{x^2-4}$ b) $\frac{x^2-5x-6}{x^2-36}$ c) $\frac{4x^2+2x-2}{4x^2+8x+6}$

6. Consider a rectangle with dimensions given by: length = $\frac{x}{x-2}$ and width = $\frac{x-2}{x^2}$

   Show that a simplified expression for the area of the rectangle is $A = 1/x$.
7. Use the given expressions for length and area, from question 6, to derive the expression for width, by applying division. Explain your solution, using mathematics combined with a written explanation.

Teacher Facilitation

- Extend the study of rational expressions to include addition and subtraction, using the concept of adding or subtracting areas in order to provide a rationale.
- Review the process of determining a common denominator CD for rational numbers, and then extend this method to rational expressions, using examples such as:
  (a) 2, 3 CD: 6 (b) 9, 3 CD: 9 (c) m, n CD: mn (d) $x, x + 1$ CD: $x(x+1)$
- Use traditional worked examples, followed by appropriate exercises, to consolidate skills in adding and subtracting rational expressions.

Assessment & Evaluation of Student Achievement

Questions 1, 6, and 7 from Sample Worksheet 1 can be used to assess Communication and Application. Parts of the rubrics used earlier in the unit can be adapted for this purpose. Refer also, to the "Generic Rubrics" for Communication and Thinking/Inquiry/Problem Solving, from the "CARE Package", which can be downloaded from http://www.oame.on.ca. A quiz can be used to assess Knowledge, after students have had opportunity to consolidate skills.

Activity 6: Power Play

Time: 75 minutes

Description

Students discover the nature of powers containing rational exponents. Students extend exponent laws to expressions involving powers containing integer and rational exponents.

Strand(s): Tools for Operating and Communicating with Functions

Overall Expectations

OCV.01 - demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;
OCV.03 - communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations

OC1.08 - simplify and evaluate expressions containing integer and rational exponents, using the laws of exponents;
OC3.01 - explain mathematical processes, methods of solution, and concepts clearly to others;
OC3.05 - use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Prior Knowledge & Skills

- Apply exponent laws to simplify expressions involving powers which contain natural exponents;
- Use the zoom and trace features of a graphing calculator to determine points on a graph.

Planning Notes

- Students will require graphing calculators.
- Prepare worksheets.

Teaching/Learning Strategies

Student Activity

Students:

- investigate the properties of powers containing rational exponents;
- apply exponent laws to simplify expressions involving powers which contain integer and rational exponents.

Teacher Facilitation

- Review with the class, the meaning of powers with natural bases, as a repeated multiplication process, using examples such as $4^3 = 4 \times 4 \times 4$.
- Next, review the meaning of powers containing a zero, or negative exponent, using examples such as $4^0 = 1$ and $4^{-2} = \frac{1}{4^2}$. Discuss the fact that these definitions of powers are not intuitively consistent with the concept of repeated multiplication.

- Pose the question: "How should we interpret powers having rational exponents, e.g., $4^{1/2}$? Considering that $4^0 = 1$ and $4^1 = 4$, should the value of $4^{1/2} = 2.5$ (arrived at through linear interpolation)?" Have students determine the value of $4^{1/2}$ using their scientific or graphing calculator.
- Have students consider that $4^2 = 16$, and hypothesize the value of $4^{3/2}$, or $4^{1.5}$, and of $4^{-1/2}$, then test their hypotheses using their calculators.
- At this point, have students graph the function $y = 4^x$, using a graphing calculator. Have them use the zoom and trace functions to verify that the values above lie on this graph. Discuss the curvature of the graph, noting the asymptotic behaviour as x approaches negative infinity.
- Challenge students to discover another way of expressing any power to the exponent $\frac{1}{2}$, i.e., $x^{1/2} = ?$ This will be revisited as students work through Sample Worksheet 1.
- Distribute Sample Worksheet 1 and have students work through the questions independently, or in small groups.

Sample Worksheet 1

1. (a) Investigate powers having different bases raised to the exponent ½.
   (b) Square the result in each case. Explain what you notice.
2. Consider that $3^2 \times 3^4$ in expanded form is $(3)(3) \times (3)(3)(3)(3) = 3^{2+4} = 3^6$

The algebraic way of generalizing this result is: $m^x \times m^y = m^{x+y}$ (*Product Rule*)

Apply the product rule to: $4^{1/2} \times 4^{1/2}$

3. Apply the product rule to: $27^{1/3} \times 27^{1/3} \times 27^{1/3}$
4. Determine a value of $27^{1/3}$, using a scientific, or graphing calculator.
5. Use the results above to give another mathematical meaning of $4^{1/2}$ and $27^{1/3}$. Explain, using words and mathematical symbols.
6. (a) Generalize the result of question 5 to explain the meaning of $x^{1/n}$. Use words and symbols.
   (b) Use numerical examples to illustrate your explanation.
7. Consider that $\frac{5^6}{5^2}$ can be expanded into $\frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5} = 5^{6-2} = 5^4$

   The algebraic way of generalizing this result is: $m^x \div m^y = m^{x-y}$ (*Quotient Rule*)
   (a) Apply the quotient rule to $6^3 \div 6^5$, and express as a power with a negative exponent.
   (b) Show that $6^3 \div 6^5 = \frac{1}{6^2}$ by writing the expression as an expanded rational expression.
   (c) What must be true about the expressions from (a) and (b)?
8. Use three methods to show that to show that $3^{-3} \div 3^{-7} = 81$
9. Use three methods to determine the value of $27^{-1/3}$
10. Explain why the graph of $y = 4^x$, asymptotically approaches the x-axis, as x approaches negative infinity.

Teacher Facilitation

- At this point, the class should reassemble to discuss their findings, and share their methods.
- The teacher should, through worked examples, consolidate concepts, and demonstrate that the "power of a power" exponent law also applies to integer and rational exponents.
- Use examples such as: a) $\frac{(12w^{-2}s^4)(15w^7s^{-11})}{20w^{-4}s^5}$ b) If $w = -1$, s = 2, evaluate $\left(w^{4/3}s^{-3/2}\right)^3\left(w^{-2}s^{5/2}\right)$

Assessment & Evaluation of Student Achievement

Questions 1, 5, and 6 can be assessed for Communication. Questions 3, 4, and 7 can be assessed for Application. Questions 9 and 10, can be assessed for Thinking/Inquiry/Problem Solving. Parts of rubrics used earlier in this unit may be adapted for this purpose. A quiz with an objective marking scheme can be used to measure Understanding, after students have had a chance to consolidate skills.

Extension
Students can explore properties of powers containing complex bases, and investigate whether or not exponent laws hold.

Activity 7: It's Snowing Cats and Dogs!

Time: 75 minutes

Description

Students extend their modelling and critical thinking skills to analyse an exponential growth pattern. Students discover that a quadratic model, which appears to effectively describe a natural phenomenon, breaks down as more data is collected, leading to the discovery of the exponential function. Students apply the properties of exponential functions to answer questions, and draw connections to, the algebraic and graphical forms of exponential relationships.

Strand(s) & Learning Expectations

Strand(s): Tools for Operating and Communicating with Functions

Overall Expectations
OCV.01 - demonstrate facility in manipulating polynomials, rational expressions, and exponential expressions;
OCV.03 - communicate mathematical reasoning with precision and clarity throughout the course.

Specific Expectations
OC1.09 - solve exponential equations (e.g., $4^x = 8^{x+3}$, $2^{2x} - 2^x = 12$);
OC3.03 - communicate solutions to problems and to findings of investigations clearly and concisely, orally and in writing, using an effective integration of essay and mathematical forms;
OC3.05 - use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve).

Prior Knowledge & Skills

- Model linear and quadratic relations using graphing technology;
- Perform finite difference analysis on data;
- Apply exponent laws.

Planning Notes

- Students require graphing calculators or access to appropriate graphing software.
- Prepare worksheets.

Teaching/Learning Strategies

Student Activity
Students:
- plot and analyse data from a simulated situation;
- discover the nature of exponential functions;
- answer questions and solve problems with involving exponential equations.

Teacher Facilitation
- Organize the class into pairs or small groups.
- Some of the techniques required for the graphing technology component of this activity may require introduction or review (e.g., generating and storing regression equations).

- Students may be familiar with the correlation coefficient, *r*, which measures how well a line fits a set of data points. When applying regression analysis to non-linear data, a more appropriate measure of best fit is the coefficient of determination, r^2. The closer r^2 is to 1, the better the curve fits the data. Since a line is simply a type of curve, r^2 can also be applied to linear data. The teacher should explain this to the class, using a couple of simple examples.
- Introduce the scenario described on Sample Worksheet 1.

Sample Worksheet 1

Suppose you are the managers of a small ski lodge, with no artificial snow production equipment. You are receiving calls early in the ski season requesting information about the skiing conditions, particularly concerning the amount of snow expected for the upcoming weekend. The weather report has alerted that a heavy snowfall is due, and is expected to follow a consistent pattern for several days. Following the first signs of snow, you measure the amount of accumulated snow, every four hours, and record the following:

Time (hours)	4	8	12	16	20	24
Amount of Snow (cm)	1.7	3.0	5.3	9.2	16	28

In order to answer customers' questions with some degree of accuracy, you decide to mathematically model this snowfall pattern, using graphing technology.

1. Plot the data from the table. Compare the pattern to functions with which you are familiar (e.g., linear, quadratic). Use the regression feature of your graphing calculator to generate:
   (a) a line of best fit,
   (b) a quadratic curve of best fit.
   For both of these, record the equation, the coefficient of determination, r^2, and a sketch of the graph. How well do these regression models fit the data?
2. Look at the first and second differences for your data. Do these satisfy the criteria for a linear or quadratic relationship? Explain your thinking.
   Suppose you record the next two measurements as follows:

Time (hours)	28	32
Amount of Snow (cm)	49	84

3. Add this data to your table and graph. How well do the best fit line and best fit quadratic relation fit the data now?
4. Experiment with different types of regression. Find an equation that fits the data with a coefficient of determination at least $r^2 > 0.999$. Record the equation and store it as y_1.
5. Consider the following two equations, which probably look different from the one you just discovered:(i) $y_2 = 4^{0.1x}$ (ii) $y_3 = 2^{0.2x}$
   (a) Graph each of these functions over your original data points. How well do these fit your data? Explain how this is possible.
   (b) Prove that y_2 and y_3 are identical functions.
   (c) Prove that y_1 is not identical to y_2 and y_3.
6. In order for successful cross-country skiing, about 30 cm of snow is required, while successful downhill conditions require about 60 cm. Suppose the snowfall started at 3:00 p.m. on Thursday. When could you recommend to your customers that skiing conditions will be satisfactory for:
   (a) cross-country? (b) downhill?

Teacher Facilitation

- Once students have had time to perform the activity above, discuss the answers to the questions.
  Sample solution to Question 5(b): $4^{0.1x} = (2^2)^{0.1x} = 2^{0.2x}$
  Sample approach to Question 5(c): show that a point, which satisfies y_1 does not satisfy y_2, by substitution.

- At this time, remove the contextual framework, and focus on skill development. Introduce methods for solving exponential equations. Use examples such as:

  a) $4^x = 64$ b) $5^{2x} = 5^4$ c) $4^{-2x} = \frac{1}{16}$ d) $2^{2x} - 2^x = 12$

Assign appropriate work from a worksheet, or the textbook, to consolidate algebraic skills.

Assessment & Evaluation of Student Achievement

Criteria	Level 1 (50-59%)	Level 2 (60-69%)	Level 3 (70-79%)	Level 4 (80-100%)
Inquiry Performs an investigation	- investigates with limited effectiveness	- investigates with moderate effectiveness	- investigates effectively	- investigates efficiently and effectively
Application Uses graphing technology effectively, (e.g., sets display window, zoom, trace, etc.)	- uses graphing technology with limited effectiveness	- uses graphing technology with some effectiveness	- uses graphing technology with considerable effectiveness	- uses graphing technology with a high degree of effectiveness
Communication Clearly explains answers to questions (oral and written) related to the activity	- explains with limited clarity	- explains with some clarity	- explains with considerable clarity	- explains with a high degree of clarity

Accommodations

Student skill in using the graphing technology may vary. It may be appropriate to pair, or group, students accordingly, as the focus of the investigation should be on the discovery of the mathematical principles.

Activity 8: Summative Assessment

Time: 75 minutes

Overall Expectations: OCV.01, OCV.03.

Specific Expectations: All expectations within unit

Description

A comprehensive, balanced summative assessment addressing all four Achievement Chart categories should be administered at the end of this unit. Three sample questions are provided which model how the teacher may assess for Communication, Application, and Thinking/Inquiry/Problem Solving. Traditional questions can be used to assess Knowledge and Understanding. These are samples only; the teacher should develop a complete summative assessment which addresses all expectations within this unit.

Planning Notes

- Graphing calculators will be required for some parts of the assessment.
- Prepare a complete, comprehensive summative assessment.

Sample Questions

1. **Sample Communication Assessment**
   Consider a quadratic function whose corresponding quadratic equation has no real roots. Explain what this information reveals about the:
   (a) x-intercepts of the graph of the quadratic function;
   (b) value of the discriminant of the quadratic equation;
   (c) direction of opening and relative position of the vertex, with respect to the x-axis, of the parabola.
   Use complete sentences, mathematical symbols and/or diagrams to explain your answers.
2. **Sample Application Assessment**
   A quadratic equation has roots $5 + 3i$ and $5 - 3i$. Determine the vertex of the graph of its corresponding quadratic function. A graphing calculator is not permitted.
3. **Sample Thinking/Inquiry/Problem Solving Assessment**
   Consider the following growth patterns of two bacteria cultures. Culture A, with an initial population of 100, doubles every hour. Culture B, with an initial population of 30, triples every hour. After what elapsed time, to the nearest minute, will the two cultures have the same population? A graphing calculator is permitted.

Unit 4: Trigonometric Functions

Time: 19 hours

Unit Description

Students investigate the periodic nature and graphical properties of the primary trigonometric functions. Using technology, students explore the effects of simple transformations on their graphs and equations. Students apply these concepts to model authentic problems.

Activity 4.1: Surf's Up! Let's Catch the Sine Wave

Time: 75 minutes

Description

Students investigate the shape of the graphs of $y = \sin x$ and $y = \cos x$ by plotting the values of these functions as determined from the unit circle.

Strand(s): Trigonometric Functions

Overall Expectations

TFV.02 - demonstrate an understanding of the meaning and application of radian measure;
TFV.03 - determine, through investigation, the relationships between the graphs and the equations of sinusoidal functions.

Specific Expectations

TF2.07 - demonstrate facility in the use of radian measure in solving and graphing equations;
TF3.01 - sketch the graphs of $y = \sin x$ and $y = \cos x$, and describe their periodic properties.

Ontario Catholic School Graduate Expectations

CGE3c - a reflective and critical thinker who thinks reflectively and creatively to evaluate situations and solve problems.

Prior Knowledge & Skills

- Students should be proficient in the use of both radian and degree measure.
- The meaning of θ and its use in representing angle measures should be introduced to the students prior to this activity.

Planning Notes

- The teacher requires an overhead projector, acetate sheet and markers, an overhead projection tablet, a graphing calculator with motion sensor attachment (such as a CBR), and a piece of string (40 to 100 cm in length) and a mass (large washers will do) to construct a pendulum.
- The teacher should also provide a unit circle, preferably superimposed on graph paper. The students require graph paper.

Teaching/Learning Strategies

This activity is comprised of two sections. The teacher should facilitate a discussion of periodic and cyclical behaviour, and demonstrate sinusoidal motion in the swinging of a pendulum. The students then derive the graphs of $y = \sin x$ and $y = \cos x$ from the unit circle.

A. Teacher Facilitation - Demonstration

- As an introduction to the graphs of sinusoidal functions, the teacher should first, by directed questioning, acquaint students with the cyclical character of the unit circle, specifically the repetitive nature of the values of $y = \sin x$ and $y = \cos x$. The students should then be asked to provide everyday examples that exhibit similar behaviour (i.e., the hands of a clock, rocking chairs, pendulums, Ferris wheels, the tides, etc.).
- As a demonstration, the teacher should set up a simple pendulum in the class and use the motion sensor to measure the position of the bob (mass) as a function of time. For best results, the mass should be pulled back about 30 cm from the rest position and the motion sensor should measure at least 4 full oscillations. Display the graph on the overhead. The graph will be sinusoidal and should give the students an indication of the type of graphs that they will be exploring throughout this unit. The periodic characteristics of the graph (specifically the amplitude, period, domain, and range) should be discussed at this point.

B. Student Activity

Suggestions for teacher facilitation are included throughout this activity in italics.

1. Convert the angles given in the table below into radian measure and enter their values into the table.
2. Starting from the positive x-axis (calling this $\theta = 0°$) and rotating counter-clockwise in 15° increments, determine the coordinates of the points on the unit circle for each angle, and enter these coordinates in the table. *Teachers may need to demonstrate how to determine co-ordinates.*
3. Determine the values of $\sin\theta$ and $\cos\theta$ for each of the given angles to two decimal places. *It is assumed that the students have been acquainted with the relationship of* $\sin\theta$ *and* $\cos\theta$ *and the unit circle in Unit 3: Trigonometry.*

θ (in degrees)	0	15	30	45	60	75	90	105	120	135	150	…	360
θ (in radians)													
coordinates (x, y)													
$\sin\theta = \frac{y}{r}$													
$\cos\theta = \frac{x}{r}$													

4. On the same set of axes, plot the graphs of $y = \sin\theta$ and $y = \cos\theta$.

C. Follow-Up Skills

Using the graphing calculator or a sample of students' work transferred to an acetate sheet, the teacher should elicit from the students pertinent characteristics of the graphs of $y = \sin\theta$ and $y = \cos\theta$. These characteristics should include the period, amplitude, roots, symmetry, domain, and range. The similarity of the graphs of $y = \sin\theta$ and $y = \cos\theta$ should also be noted, and in particular the translation that would map one graph onto the other. The teacher should introduce the term *phase shift* at this point.

Assessment & Evaluation of Student Achievement

Learning skills can be assessed visually as the students are completing their graphs. Their ability to work independently can be assessed using as criteria accomplishing a task independently and self-direction. Time management skills can be used to assess organization. Initiative can be assessed using the students' self-motivation and responses to prompts by the teacher as criteria.

Activity 4.2: Transformations of Trigonometric Functions

Time: 150 minutes

Description

Students investigate the effects of simple transformations on the graphs of $y = \sin x$ and $y = \cos x$ through the use of graphing technology.

Strand(s): Trigonometric Functions, Tools for Operating and Communicating with Functions

Overall Expectations

TFV.02 - demonstrate an understanding of the meaning and application of radian measure;
TFV.03 - determine, through investigation, the relationships between the graphs and the equations of sinusoidal functions;
OCV.02 - demonstrate an understanding of inverses and transformations of functions and facility in the use of function notation.

Specific Expectations

TF2.07 - demonstrate facility in the use of radian measure in solving and graphing equations;
TF3.02 - determine, through investigation, using graphing calculators or graphing software, the effect of simple transformations (e.g., translations, reflections, stretches) on the graphs and equations of $y = \sin x$ and $y = \cos x$;
TF3.03 - determine the amplitude, period, phase shift, domain, and range of sinusoidal functions whose equations are given in the form $y = a \sin(kx + d) + c$ or $y = a \cos(kx + d) + c$;
TF3.04 - sketch the graphs of simple sinusoidal functions [e.g., $y = a \sin x$, $y = \cos kx$, $y = \sin(x + d)$, $y = a \cos kx + c$];
TF3.05 - write the equation of a sinusoidal function, given its graph and given its properties;
OC2.06 - represent transformations (e.g., translations, reflections, stretches) of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$, using function notation;
OC2.07 - describe, by interpreting function notation, the relationship between the graph of a function and its image under one or more transformations;
OC2.08 - state the domain and range of transformations of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$.

Ontario Catholic School Graduate Expectations

CGE3c - a reflective and critical thinker who thinks reflectively and creatively to evaluate situations and solve problems;
CGE5a - works effectively as an interdependent team member;
CGE5e - respects the rights, responsibilities and contributions of self and others;
CGE5f - exercises Christian leadership in the achievement of individual and group goals.

Prior Knowledge & Skills

- Students should be proficient in the use of radian measure.
- The graphs and characteristics of $y = \sin x$ and $y = \cos x$ should be familiar to all students.
- Students should be proficient in the use of graphing calculators to plot functions. Students should also be able to graph manually.
- The meaning of θ and its use in representing angle measures should be introduced to the students prior to this activity.

Planning Notes

- The teacher requires an overhead projector, transparencies, and markers. (An overhead projection tablet could be used as well.) In addition, the teacher must prepare a student worksheet for this activity.

- The students require graphing calculators (or dynamic graphing software such as *Zap-A-Graph* or *Geometer's Sketchpad*), and graph paper.
- The first part of the activity is a discovery exercise, after which the students summarize their results. The second part of the activity includes a similar investigation involving more complex transformations.
- This activity may also be done without the use of technology, if required.

Part 1: Transformations: More than Meets the Eye

Time: 75 minutes

Teaching/Learning Strategies

A. Teacher Facilitation

- Students are placed into groups of two or three. Each group is assigned a set of four particular examples of sinusoidal functions of the following types:

Group 1	$y = a \sin x$	Group 2	$y = a \cos x$
Group 3	$y = \sin kx$	Group 4	$y = \cos kx$
Group 5	$y = \sin(x + s)$	Group 6	$y = \cos(x + s)$
Group 7	$y = \sin x + c$	Group 8	$y = \cos x + c$

- This activity is described for Group 1 only. Groups 2 to 8 answer similar questions using a variety of values for the given parameters. In particular, it is recommended that both positive and negative values be used, and that both whole numbers and fractions are included. For example, Group 3 might be assigned the functions $y = \sin 3x$, $y = \sin(-\frac{1}{4}x)$, $y = \sin \frac{2}{3}x$, and $y = \sin(-4x)$.
- As trigonometric functions are traditionally considered to be functions of radian measure, it should be noted that the values given for the constant s (Groups 5 and 6) should be multiples of π.
- Students are asked to describe the graphs of the assigned functions. The teacher should expect these descriptions to include at least the maximum and minimum points, the amplitude, the roots, the period, symmetry, the phase shift, and the domain and range.

B. Student Activity

Suggestions for teacher facilitation are included throughout this activity in italics.

1. a) Plot a graph of $y = \sin x$ on the graphing calculator, using a window large enough to display two complete cycles, one on either side of the y- axis. Reproduce this sketch on the graph paper.
   b) Graph the following curves on the calculator, stopping after each to reproduce the plot on the graph paper, using the same set of axes as in question 1a: $y = 2 \sin x$, $y = -\sin x$, $y = \frac{2}{5} \sin x$, and $y = -\frac{1}{2} \sin x$.
   c) In what ways are the graphs similar? In what ways are they different?
2. a) Clear the screen, then plot a graph of $y = \cos x$ on the graphing calculator, using a window large enough to display two complete cycles, one on either side of the y- axis. Reproduce this sketch on the graph paper using a new set of axes.
   b) Use the calculator to graph each of the functions $y = -3 \cos x$, $y = 5 \cos x$, $y = \frac{1}{4} \cos x$, and $y = -\frac{1}{3} \cos x$. After graphing each equation, reproduce the plot on the graph paper, using the same set of axes as in question 2a.
   c) In what ways are the graphs similar? In what ways are they different?
   d) Compare and contrast these graphs to those sketched in question 1.

3. Without plotting, compare and contrast the graphs of the following equations:
   a) $y = \sin x$ and $y = 12 \sin x$ b) $y = \sin x$ and $y = -\frac{1}{5} \sin x$ c) $y = \cos x$ and $y = -3 \cos x$

After each group has completed their respective investigation, they present their results to the rest of the class, drawing attention to the role of the constants in each transformation, and thus making connections between the function and its graph.

C. Follow-up Skills
Time: 30 minutes
The teacher should assign additional questions to the students in class to reinforce the new learning. The teacher could facilitate a class discussion to confirm responses, and use the opportunity to provide remediation as necessary. The given questions should include a variety of constants, as outlined above.

Part 2: Give Me A Sine
Time: 75 minutes

Teaching/Learning Strategies
A. Teacher Facilitation
- The second part of the activity proceeds in much the same manner as the first. The following sinusoidal functions will be assigned to each group.

Group 1	$y = a \sin kx$	Group 2	$y = a \cos kx$
Group 3	$y = a \sin(x + s)$	Group 4	$y = a \cos(x + s)$
Group 5	$y = \sin kx + c$	Group 6	$y = \cos kx + c$
Group 7	$y = \sin k(x + s)$	Group 8	$y = \cos k(x + s)$

B. Student Activity
Once again, each group is asked to present their results to the rest of the class, drawing attention to the effect of each numerical value in the given functions.

C. Follow-up Skills
Time: 45 minutes
To prepare students for subsequent activities, the teacher should further explore some of the concepts learned in this activity:
- Functions of the form $y = \sin(x + s)$ and $y = \cos(x + s)$ have, until now, included only multiples of π for the value of s. The teacher should have students explore the effect of other real values of s, as they play a most important role in the modelling of authentic problems later in this unit.
- A connection must be established between the functions $y = \sin k(x + s)$ (studied in this activity) and $y = \sin(kx + d)$ (an alternate form), and between the functions $y = \cos k(x + s)$ and $y = \cos(kx + d)$. Specifically, it is imperative that students recognize the roles of k, s, and d in determining the period and the phase shift of a given function. Both of these forms are examined in subsequent activities, so students should be comfortable working with them both.
- In this activity, the students are asked to compare and contrast the equations and graphs of simple transformations of $y = \sin x$ and $y = \cos x$. The teacher should provide additional questions to reinforce these skills. In addition, students should be asked to sketch a variety of given sinusoidal functions, and to determine the equation of various sinusoidal functions given their graphs. These skills are of great importance to achieve success in the rest of this unit.
- Suitable textbook questions should be assigned to consolidate these concepts.

Assessment & Evaluation of Student Achievement

The students' oral reports may be assessed using an appropriate group presentation rubric, with emphasis paid to the assessment of Communication skills, particularly the use of mathematical symbols and conventions. Other criteria may include the computation and construction of graphs (Knowledge and Communication), and reasoning skills (Inquiry and Communication). After completing the recommended textbook exercise, a short pencil-and-paper task could be used to assess the core knowledge and concepts. Learning skills, specifically initiative, could be assessed as the teacher facilitates the classroom discussion to conclude the first part of the activity. Teamwork skills can be assessed visually as the groups complete their investigation. The students' work habits and organization can be assessed during their presentations.

Accommodation

The teacher should ensure that those students with difficulties in understanding concepts be placed into groups in which they can receive support from other students.

Activity 4.3: Don't Go Off on a Tangent

Time: 75 minutes

Description

Students identify the defining characteristics of the function $y = \tan x$, and establish connections with the functions $y = \sin x$ and $y = \cos x$.

Strand(s): Trigonometric Functions

Overall Expectations

TFV.03 - determine, through investigation, the relationships between the graphs and the equations of sinusoidal functions.

Specific Expectations

TF3.01 - sketch $y = \sin x$ and $y = \cos x$, and describe their periodic properties;
TF3.06 - sketch the graph of $y = \tan x$; identify the period, domain, and range of the function; and explain the occurrence of asymptotes.

Ontario Catholic School Graduate Expectations

CGE3c - a reflective and creative thinker who thinks reflectively and creatively to evaluate situations and solve problems;
CGE4b - a self-directed, responsible, life long learner who demonstrates flexibility and adaptability;
CGE4f - a self-directed, responsible, life long learner who applies effective communication, decision-making, problem-solving, time and resource management skills.

Prior Knowledge & Skills

- Students should possess a comprehensive knowledge of the division of rational numbers, and specifically the concept of division by 0.
- The primary trigonometric ratios should be familiar to all students.
- Students should understand the concept of an asymptote, as defined in Unit 1: Algebraic Manipulation of Functions and Unit 2: Function Notation, Inverses, and Transformations.
- Students must be able to use the radian and degree modes of a scientific calculator.

Planning Notes

- Students must have graph paper and a scientific calculator for use in the graphing of the primary trigonometric functions.
- The teacher has to prepare the tables to be filled out by the students in this activity.

Teaching/Learning Strategies

A. Teacher Facilitation

- Students use characteristics of the graphs of $y = \sin x$ and $y = \cos x$ and the relationship of the primary trigonometric ratios to determine the graph of $y = \tan x$, after which they make observations about its properties.
- The teacher may pair or group students for this activity, or they may have the students work independently.
- As the class is completing this investigation, the teacher should be addressing students' concerns individually or, if necessary, in small groups. In addition, students should be encouraged to seek help from other students.

B. Student Activity

Suggestions for teacher facilitation are included throughout this activity in italics. Some solutions are included to aid in the flow of the activity.

1. Find the missing angle measures in the given table.
2. Using a scientific calculator, determine the values of sin *x* and cos *x* for the given angles to the nearest thousandth.

x (in radians)		$\frac{\pi}{12}$	$\frac{\pi}{6}$			$\frac{5\pi}{12}$		$\frac{7\pi}{12}$		$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	…	
x (in degrees)	0			45	60		90		120			…	360
sin *x*													
cos *x*													
sin *x* ÷ cos *x*													
tan *x*													

3. Use the values in the table to sketch the graphs of $y = \sin x$ and $y = \cos x$.
4. List as many characteristics as possible about these graphs. Students should include in their list the domain, the range, the roots, the period, and the amplitude.
5. Using the values found in question 2, calculate the ratio $\frac{\sin x}{\cos x}$ to the nearest thousandth, and enter the results in the table.
6. Use the calculator to compute values for tan *x* to the nearest thousandth, using the angles given in the table. Compare these values with those found in question 5.
7. Plot the values of tan *x*. There may be some difficulties for certain values of *x*. Why? The graph of $y = \tan x$ is said to have an *asymptote* at these points, and is to be represented on the graph by a dotted vertical line. What happens to the graph as the angle values approach these asymptotes? *Some values of x (90°, for example) produce undefined answers for y* = tan *x. The concept of an asymptote is introduced in Unit 1: Algebraic Manipulation of Functions and Unit 2: Function Notation, Inverses, and Transformations.*
8. Sketch the graph of $y = \tan x$, including all asymptotes.
9. Describe the graph of $y = \tan x$, calling attention to as many characteristics as possible. Any description should include the following information: domain is {x ∈ R, 0 # x # 360°, $x \neq 90°$, 270°}, range is $\{y \in R\}$, asymptotes at $x = 90°$ and $x = 270°$, graph repeats itself every 180° (the period), roots are $x = 0°$, $x = 180°$, $x = 360°$.

10. Using the primary trigonometric ratios, prove that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
11. Using the table and the graph, predict values for the following:
    a) $\tan 420°$ b) $\tan(-\frac{5\pi}{12})$ c) $\tan(-135°)$ d) $\tan\frac{8\pi}{3}$.
    Confirm these values with a scientific calculator.

C. Follow-up Skills
Time: 30 minutes
The teacher may wish to facilitate a classroom discussion incorporating some of the results of this investigation. The fact that tan *x* can be defined as the quotient of sin *x* and cos *x* is significant, as are the properties that $y = \tan x$ has the same roots as $y = \sin x$, and undefined values for the roots of $y = \cos x$. The behaviour of $y = \tan x$ around its asymptotes should also be reviewed. Suitable textbook exercises should be used to reinforce concepts introduced in this investigation. The teacher can then remediate as necessary.

Assessment & Evaluation of Student Achievement
This investigation can be used to assess the students' independent work skills or teamwork skills, particularly their ability to stay on task. The teacher can use an appropriate observational rubric to assess the students' progress with the mathematical content. The teacher may have the students submit questions 7, 8, and 9 as a journal topic. Questions 7 and 8 address Knowledge and Inquiry. Assessment criteria could include the accuracy of calculations, the justification of the asymptotes, and the accuracy of the graph. Question 9 emphasizes Knowledge and Communication, specifically the use of mathematical vocabulary and symbols, and the inclusion of all desired defining characteristics.

Activity 4.4: Applications of Trigonometric Functions
Time: 450 minutes

Description
Both natural phenomena and manufactured devices exhibit sinusoidal motion. Rotations, oscillations, and waves all exhibit sinusoidal motion. In three different activities, the motion of a spring, a bicycle wheel, and the sun are used to connect the transformations of trigonometric functions to the practical world. During these investigations, students become more familiar with the manipulation and properties of the general sine function.

Strand(s): Trigonometric Functions, Tools for Operating and Communicating with Functions

Overall Expectations
TFV.03 - determine, through investigation, the relationships between the graphs and the equations of sinusoidal functions;
TFV.04 - solve problems involving models of sinusoidal functions drawn from a variety of applications;
OCV.02 - demonstrate an understanding of inverses and transformations of functions and facility in the use of function notation.

Specific Expectations
TF3.02 - determine, through investigation, using graphing calculators or graphing software, the effect of simple transformations, (e.g., translations, reflections, stretches) on the graphs and equations of $y = \sin x$ and $y = \cos x$;
TF3.03 - determine the amplitude, period, phase shift, domain, and range of sinusoidal functions whose equations are given in the form $y = a\sin(kx + d) + c$ or $y = a\cos(kx + d) + c$;

TF3.04 - sketch the graphs of simple sinusoidal functions (e.g., $y = a \sin x$, $y = \cos kx$, $y = \sin (x + d)$, $y = a \cos kx + c$);
TF3.05 - write the equation of a sinusoidal function, given its graph and given its properties;
TF4.01 - determine, through investigation, the periodic properties of various models (e.g., the table of values, the graph, the equation) of sinusoidal functions drawn from a variety of applications;
TF4.02 - explain the relationship between the properties of a sinusoidal function and the parameters of its equation, within the context of an application, and over a restricted domain;
OC2.06 – represent transformations (e.g., translations, reflections, stretches) of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$, using function notation;
OC2.08 – state the domain and range of transformations of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$.

Ontario Catholic School Graduate Expectations
CGE 2c - an effective communicator who presents information and ideas clearly and honestly and with sensitivity to others;
CGE 3c - a reflective and creative thinker who thinks reflectively and creatively to evaluate situations and solve problems;
CGE 4b - a self-directed, responsible, life long learner who demonstrates flexibility and adaptability;
CGE 5a - a collaborative contributor who works effectively as an interdependent team member;
CGE 5g - a collaborative contributor who achieves excellence, originality, and integrity in one's own work and supports these qualities in the work of others.

Prior Knowledge & Skills

- Students should possess competent manual graphing skills, and be proficient in the use of a graphing calculator or computer software specifically in the compiling and graphing of data.
- Students must be able to collect data using a variety of methods (motion sensor/sensory probe, research, measurement).
- Students should possess a thorough understanding of the general properties of sinusoidal functions and their graphs.
- Radian measure, as it pertains to trigonometric equations, should be familiar to all students.

Planning Notes

- These activities have been developed with the intention that they would be conducted sequentially. It is possible, however, for them to be conducted concurrently, with groups of students rotating through each of them.
- The activities differ enough that they may cater to the various learning styles and abilities present in the class. Activity 4.4B is probably the most tactile, and Activity 4.4C is probably the most abstract. The teacher may wish to assign students to a specific activity.

Activity 4.4A: It's a Spring Thing

Time: 150 minutes

Description

Students study the motion of a mass on a spring using a graphing calculator with a motion sensor attachment.

Planning Notes

- As students are placed into groups for this activity, the teacher should ensure that there are enough graphing calculators and motion sensor units for each group. Each group should have an acetate sheet, a spring holder, a given mass and a different spring.
- This activity has strong ties to several topics in physics, particularly Hooke's Law. The teacher may wish to consult and collaborate with the physics teacher in order to emphasize the cross-curricular tie.

Teaching/Learning Strategies

A. Teacher Facilitation

- Students should be placed in groups of two or three.
- When collecting data with the motion sensor, the calculator should NOT be in *realtime* mode, and should be set long enough to record at least four cycles. The calculator should also be set for *heavy* smoothing to make the graphs easier to match.
- The mass should not be closer than 50 cm to the motion sensor at any point during the collection of data.
- The fact that the groups have different springs gives each group different periods. If the teacher does not have different springs, a similar result can be achieved if each group has the same spring but different masses.
- The results of this investigation are recorded on acetate sheets. Once each group has completed the investigation, these acetate sheets should be discussed in class.
- This investigation uses the general sine function $y = a \sin(kx + d) + c$, which differs from other forms studied thus far in this unit (refer to Activity 4.2 – Follow-Up Skills).

B. Student Activity

Suggestions for teacher facilitation are included throughout this activity in italics. Some solutions are included to aid in the flow of the activity.

1. Set up the apparatus such that the spring and mass are secure and oscillate freely (refer to Figure 1, below). Measure the distance between the motion sensor and the mass and call this the Equilibrium Position.
2. Start the spring in motion by pulling it down approximately 10 cm from the Equilibrium Position.
3. Once the spring is in motion, begin collecting data using the appropriate program on the graphing calculator.
4. Once the data is collected and plotted, it should take the form of a sinusoidal curve. If it does not, resample the data.
5. As the motion sensor measures the position of the mass over time, the time data will automatically be stored in L_1, and the distance data in L_2.
6. Using previous knowledge of transformations, determine the values of a, c, d, and k so that the graph of $y = a \sin(kx + d) + c$ matches the plot created by the motion sensor. *For this form of the curve, a is the amplitude, c is the vertical translation,* $d \div k$ *is the phase shift, and* $2\pi \div k$ *is the period.*
7. Using the data collected, determine the average position of the mass, and the difference between the highest and lowest positions of the mass. Compare these values to a, c, d, and k. *Here the average position should be approximately c, and the difference should be approximately 2a.*
8. Find the time between successive maxima, and the time between successive minima. Average these values. What is this average called? Multiply this average with the value of k, and explain the significance of the result. *The values are averaged in order to produce a more accurate measurement. The average value is called the period. Multiplying the period and k yields a number approximately equal to* 2π*, which should be so, because mathematically the period can be calculated using the formula* $p = 2\pi \div k$.

9. By using the same values for *a, c, d,* and *k*, graph the function $y = a\cos(kx + d) + c$. Discuss the similarities and differences between the sine and cosine graphs. *The function* $y = \cos x$ *is the same as the function* $y = \sin x$*, but shifted to the left by* $\frac{\pi}{2}$ *units (i.e.,* $\cos x = \sin(x + \frac{\pi}{2})$*).*

   So $a\cos(kx + d) = a\sin(k(x + \frac{\pi}{2}) + d)$*, or* $a\sin(kx + d + \frac{k\pi}{2})$*. Thus, the cosine function is the same as the sine function, but shifted to the left by* $\frac{k\pi}{2}$ *units.*
10. Repeat the experiment by pulling the spring a different distance from the Equilibrium Position.
11. How do the values of *a*, *c*, *d*, and *k* compare to those found in the first trial? *In subsequent trials the value of the constant a should be different (bigger or smaller depending on how the pull of the spring compares to the first). There should be no correspondence for the value of d, since it will depend on the initial position of the mass when the data began to be sampled. Both k and c should be the same.*
12. Discuss what aspect of the experiment controls the value of *d*.
13. Summarize the data neatly and concisely on an acetate sheet.

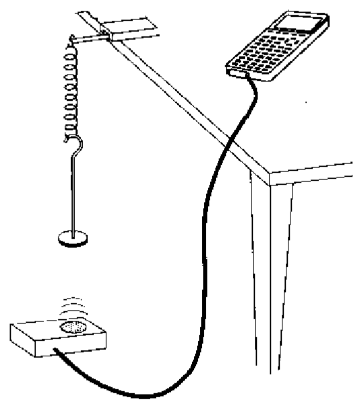


Figure 1

Assessment & Evaluation of Student Achievement

Observing and conferencing can be used to assess the students' Knowledge and Understanding while the activity is in progress. Testing different values of *a*, *c*, *d*, and *k* to obtain a theoretical equation to match the data demonstrates Inquiry/Problem Solving skills. The students' acetate sheets can be used to assess Communication skills, and should form the basis of a classroom discussion facilitated by the teacher to confirm and summarize results. The teacher may wish to assign tasks to each member of a group and assess students on the performance of their task. Sample tasks may include "copy producer" (scribe, assessed on written Communication skills), "experiment engineer" (performs experiment, assessed on Knowledge and Inquiry skills), "public relations specialist" (representative to explain results, assessed on oral Communication skills), etc.

Activity 4.4B: Ferris Fair

Time: 150 minutes

Description

Students study the height of a rider in a gondola on a Ferris wheel as the wheel rotates. To simulate the rotation of a Ferris wheel, a bicycle wheel is rolled along the ground.

Planning Notes

- Students are placed into groups for this activity. A bicycle wheel (or any other circular object, such as a hula-hoop, a paint can, etc.) should be provided to each group to simulate a Ferris wheel.
- Tape and metre sticks are needed to measure the height of the rider.

Teaching/Learning Strategies

A. Teacher Facilitation

- Students should be put into groups of two or three.
- Be sure each group has sufficient room to roll their wheel so that its motion is not obstructed.
- The teacher should note that when students are asked to compare experimental values to theoretical values, they should not only be looking for values that are similar, but also values that are multiples of each other.
- As an alternative to using a physical model, the Resources section includes a link to a website to simulate a Ferris wheel using Geometer's Sketchpad.
- The Ferris wheel could also be modelled by placing a bicycle upside down and marking one of its wheels with a piece of chalk to represent the rider.

B. Student Activity

Suggestions for teacher facilitation are included throughout this activity in italics. Some solutions are included to aid in the flow of the activity.

1. Place one piece of tape on the ground (starting point) and one on the rim of the wheel (the rider). Roll the wheel away from the starting point (see Figure 2, above) and stop the wheel in at least 16 different positions, including points at which the rider is at the top of the wheel and bottom of the wheel. At each position, measure the distance from the starting point to the bottom of the wheel (x) and the height (h) of the rider. Care should be taken to make sure that the wheel rolls in a straight line, and that the wheel makes at least one complete rotation. *If the top and bottom positions are not included, the students may get an inaccurate measurement for the amplitude.*
2. Repeat part 1 three more times, using different starting positions for the rider each time.
3. Create a table of values and plot the graph of the distance (x) vs. height (h) for each set of data, using a graphing calculator or graphing software.
4. List as many characteristics as possible of each graph. *Characteristics should include the fact that the graph is sinusoidal, the number of maxima and minima, the difference between the heights of the maxima and minima, the difference between successive maxima, the difference successive minima, the period, amplitude, phase shift, and vertical translation.*
5. Determine the values of a, c, d, and k in the equation $y = a \sin(kx + d) + c$ that models the data and matches the graph.
6. Determine the radius of the wheel. How does this value relate to the equation? *The radius should correspond to the value of a. For this investigation it will also correspond to the value of c.*
7. Find the circumference of the wheel. What is the significance of this value? *Students should recognize this value as being the period of the function. Thus, this value should be equal to* $2\pi \div k$.
8. Describe how this model of a Ferris wheel differs from a real Ferris wheel. For this model, what does the value of c correspond to? For a real Ferris wheel, would the value of c correspond to the same thing? Explain. *For a real Ferris wheel, the value of c corresponds to the distance from the ground to the centre of the wheel. This result may be better visualized using the inverted bicycle model (see Teacher Facilitation). Setting the bicycle on the floor or the desk will produce different values of c, but the value of a will remain constant.*
9. What is the significance of the value of d? *Different values of d occur depending on the starting position of the gondola.*
10. Using the same data, model the movement of the rider using the general cosine function $y = a \cos(kx + d) + c$. Answer questions 5 to 9 again, using this new equation for reference.

11. Compare the equations and graphs of the sine and cosine functions, stating any similarities and differences. *The curves are exactly the same, except for the fact that the cosine function is shifted to the left. The teacher may need to remind students that the standard cosine function* $y = \cos x$ *is shifted to the left by* $\frac{\pi}{2}$ *units from the standard sine function* $y = \sin x$*. With a little facilitation, the students should be able to determine that their cosine function is shifted to the left by* $k \times \frac{\pi}{2}$ *units.*

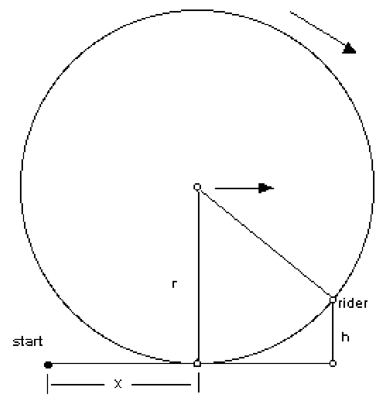


Figure 2

C. Follow-up Skills
Time: 75 minutes
The teacher should supplement the preceding activities with textbook exercises. Types of questions that should be assigned include:

- Given the graph of a sinusoidal function, determine its equation.
- Given a general sinusoidal equation of any form (i.e., $y = a \sin(kx + d) + c$, $y = a \sin k(x + s) + c$, $y = c + a \sin k(x - s)$, etc.), describe and sketch the graph of the equation by hand.

The teacher should also review the role of all of the constants in the various forms of the sinusoidal equation.

Assessment & Evaluation of Student Achievement

As in Activity 4.3A, students continue to develop confidence in the manipulation of sinusoidal equations. It is suggested that at this point that Knowledge/Understanding be assessed by a paper-and-pencil task, such as a quiz.

Activity 4.4C: Let the Sine Shine In

Time: 150 minutes

Description

Students collect sunrise and sunset data and use it to model an hours of daylight function for a particular location.

Planning Notes

- Sunrise and sunset data for thousands of locations around the world is readily available at the U.S. Naval Observatory website, http://aa.usno.navy.mil/AA/. If students have access to a computer lab, the teacher may wish to direct students to the website to collect their own data. Alternatively, students that have access to a computer at home may collect data for the class.
- Graph paper should be made available to the students, as two sets of axes are required.
- This activity has strong ties to several topics in astronomy. Suggested topics for further student research are included throughout this activity.

Teaching/Learning Strategies

A. Teacher Facilitation

- Sunrise and sunset times for any location can be calculated if its longitude and latitude are known. In order to make meaningful conclusions from the data, the sunrise and sunset times are obtained for five different cities, located at approximately 20°N, 30°N, 40°N, 50°N, and 60°N.
- The students form groups and each is assigned a latitude.
- Groups may use an atlas, if available, to choose a city, or they may use the following suggestions: 60°N - St. Petersburg (Leningrad), Russia; 50°N - Winnipeg, Canada; 40°N - Philadelphia, U.S.A.; 30°N - Cairo, Egypt; 20°N - Santiago, Cuba.
- This activity uses the general sine function $y = c + a \sin k(x - s)$. Increased exposure to different forms will help the students develop confidence in their algebraic manipulation skills.

B. Student Activity

Suggestions for teacher facilitation are included throughout this activity in italics. Some solutions are included to aid in the flow of the activity.

Part 1 – Table for Sun, Please

After obtaining the sunrise and sunset data for the given location, complete the following table. To make future calculations easier, use 24-hour metric time, (e.g., 4:00 p.m. would be 16:00). **Note:** In the chart, Sunrise [:] refers to the time in the standard format of hrs:min (e.g., 10:45), while Sunrise [.] refers to the time in decimal format (e.g., 10.75). *The first couple of rows of the proposed table would appear as follows:*

City:			Latitude:		Longitude:	
Date	Day of Year	Sunrise [:]	Sunrise [.]	Sunset [:]	Sunset [.]	Hours of Daylight [.]
01 01	0					
01 15	14					
…	…					

Note that January 1 is designated as day 0, to aid in the construction of an equation. Include in the table March 20, June 21, September 23, and December 21, and at least two additional days from each month.

Part 2 – Can You See the Light?

The intent of this section is to have the students determine sinusoidal equations by inspection. Using graphing techniques, students identify defining characteristics of a sinusoidal curve, and use them to determine its equation. It is anticipated that not all students will have access to graphing calculators. Several questions have thus been presented in two formats, one for use with graphing calculators (denoted [C]*), the other with paper-and-pencil (denoted* [P]*).*

1. If it has not already been done, compute and insert the number of hours of daylight (in decimal format) into the last column.
2. [C] Construct a scatter plot by entering the day of the year in L_1, and the hours of daylight in L_2, using a window large enough to display all of the data.
   [P] Construct a scatter plot of the day of the year vs. the hours of daylight on the first set of axes.
3. Visually estimate the amplitude, period, phase shift, and vertical translation of this graph.
4. Using the general form $h(t) = c + a \sin k(t - s)$, determine an equation that models this data. *This is yet another form of the general sinusoidal equation. The constants c, a, and s will represent the vertical translation, amplitude, and phase shift, respectively. The constant k is 360° (or 2π) divided by the period.*
5. [C] Store this equation in Y_1 and plot its graph using the dashed line. How well does it match the scatter plot?
   [P] Using a table of values, graph this curve on the same set of axes. How well does it match the scatter plot?

Part 3 – Straining Functions Through a Calendar

Students use the table to algebraically determine the equation of the curve.

1. a) What are the longest and shortest days of the year (i.e., what days receive the most and the least amount of daylight)? Estimate the length of the longest day and the shortest day. What is the range of daylight hours of the course of the year? *The longest and shortest days of the year are June 21 and December 21.*
   b) What two days receive an equal amount of daytime and night-time? *The days of equal daytime and night-time are March 20 and September 23.*
   c) Explain the significance of these four days. *These four days are used to indicate the changing of the seasons.*
2. Estimate the daily average number of hours of daylight over the entire year.
3. From previous knowledge, it can be easily verified that $y = \sin x$ passes through the origin. In order to match the hours of daylight function to the general function $y = \sin x$, where should the origin be placed? *The function $y = \sin x$ passes through the origin, which can be considered the "middle" or "average" value of the graph. To match the hours of daylight function to the sine function a similar point should be found. The origin should be placed at March 20, because this day serves as a "middle" or "average" value for the hours of daylight function (having an equal amount of daytime and night-time). This question will determine the phase shift.*
4. Estimate the period of the daylight curve. Justify your answer.
5. Using the answers to questions 1 through 4, determine the equation that can be used to model the number of hours of daylight. *Question 1 relates to the amplitude, question 2 will provide the vertical translation, question 3 will be used to determine the phase shift, and question 4 will provide the period. Incidentally, the sinusoidal regression function on most graphing calculators would be able to compute this equation.*
6. [C] Store this new equation in Y_2 and plot its graph using the thin solid line. Compare this graph with the graph plotted in part 2. By what method (visual or computational) was the best fit achieved?
   [P] Graph the new equation on the same set of axes as the previous graph using a table of values. Compare the two graphs. By what method (visual or computational) was the best fit achieved?

Part 4 – Dare to Share and Compare

For this part of the activity, the students collect data from two other groups in order to compare results. They need their second set of axes for this section.

1. [C] Clear Y_1 from the list of equations. Keep the second equation.
   [P] Replot the second graph on a new set of axes.
2. Obtain the hours of daylight data from two other groups. Take special note of the location of their chosen city.

3. Find the amplitude, period, phase shift, and vertical translation of both sets of data, and determine their respective equations.
4. [C] Store the two equations in Y_3 and Y_4. Graph the equations using the dashed line and the thick solid line, respectively, to make it easy to identify each graph.
   [P] Graph the equations on the same set of axes. Be sure to label your plots.
5. Compare and contrast the three equations.
6. Compare and contrast the three graphs. For what reasons would the graphs be similar or different?

C. Follow-up Skills

Time: 60 minutes

Supplemental textbook exercises could be used to reinforce learning. In addition, the following questions could be used to consolidate concepts and further understanding of this particular model.

1. For an hours of daylight curve of the form $h(t) = c + a \sin k(t - s)$, how were a, c, k, and s determined?
2. What is the least amount of data needed to determine a daylight curve? *Two points: the maximum and minimum points will provide the amplitude and phase shift. The vertical translation is always 12 hrs, and the period is always 1 year.*
3. Hours of daylight functions can also be modelled using cosine functions. In what ways would this function differ from the sine equation? Determine the cosine equation that could be used to model the data in the table found in Part 1 – Table for Sun, Please.

D. Supplemental Research

The following questions provide the teacher with some topics for supplemental student research.

1. Find other cities with the same latitude as those given in this activity. Would you expect the hours of daylight curves to be different or similar? Confirm your prediction with some research.
2. The longest and shortest days of the year and the days of equal daytime and night-time have special names. What are they called and why? *The longest and shortest days of the year are called the vernal and autumnal equinoxes, respectively. The days of equal daytime and night-time are called the winter and summer solstices.*
3. Can hours of daylight data be modelled as a sinusoidal function for every location on earth? Explain. *Latitudes north of the Arctic Circle or south of the Antarctic Circle experience extended periods of 24-hour daytime and night-time, hence the hours of daylight function cannot be modelled using a sinusoidal curve.*
4. From your original table, graph the sunrise and sunset data on the same set of axes using the given days as your points of reference. Are these curves sinusoidal? For any curves that do not look sinusoidal, describe the way in which it fails to be sinusoidal. Why would these curves not be sinusoidal? *Due to the tilt of the earth's axis, sunrise and sunset curves are skewed, and cannot be modelled using sinusoidal functions*.
5. Find other natural phenomena that can be modelled using sinusoidal functions.
6. Longitude and latitude are measured in degrees, minutes, and seconds. This is quite similar to our measurement of time. Why is this so? *Both the measurement of time and the measurement of longitude and latitude derive from the ancient Babylonian number system, which was base-60*.

Assessment & Evaluation of Student Achievement

All knowledge and skill categories can be assessed in this activity. Parts 2 and 3 emphasize Knowledge and Application skills, particularly in the modelling of functions using information collected by the students themselves. Part 4 focuses primarily on Knowledge and Communication skills, specifically the students' ability to compare and contrast graphs and equations. The follow-up section contains some questions requiring students to use their Inquiry, Application, and Communication skills. Criteria for assessment would include the ability to hypothesize and justify reasoning, and the ability to apply their knowledge in an unfamiliar setting. Learning skills, particularly initiative, organization, and teamwork, can be assessed using appropriate rubrics. Part 4 of the activity lends itself to group presentations, which can be assessed using a suitable oral report rubric. The follow-up and extension questions could be assigned as journal topics.

Accommodations

Because of the long list of instructions in this activity, the teacher should ensure that students with comprehension or communication difficulties are grouped with students that can assist them.

Resources

Antinone, L., S. Gough, and J. Gough. *Modeling Motion: High School Math Activities with the CBR*. Austin, TX: Texas Instruments, 1997. ISBN 1-886309-14-0

Data Services (http://aa.usno.navy.mil/AA/)

The Astronomical Applications Department of the U.S. Naval Observatory produces almanacs, software, and web services to provide precise astronomical data for practical applications, serving the defence, scientific, commercial, and civilian communities.

Ferris Wheel Rides (http://curry.edschool.virginia.edu/curry/centers/partnership/honalg2.htm)

This is a Ferris wheel demonstration that includes some sample data as well as a suggested rubric.

Modelling a Ferris Wheel Using Translations and Animation (http://mathforum.com/dynamic/jrk/ferris_dir/)

This is a tutorial to create a Ferris Wheel animation using *Geometer's Sketchpad*.

Activity 4.5: Summative Assessment

Time: 150 minutes

Description

Students demonstrate their ability to apply the skills and knowledge acquired in this unit. A summative assessment is used to determine how the students have met the expectations of this unit.

Strand(s): Trigonometric Functions, Tools for Operating and Communicating with Functions

Overall Expectations

TFV.02 - demonstrate an understanding of the meaning and application of radian measure;
TFV.03 - determine, through investigation, the relationships between the graphs and the equations of sinusoidal functions;
TFV.04 - solve problems involving models of sinusoidal functions drawn from a variety of applications;
OCV.02 - demonstrate an understanding of inverses and transformations of functions and facility in the use of function notation.

Specific Expectations
TF2.07 - demonstrate facility in the use of radian measure in solving equations and in graphing;
TF3.01 - sketch $y = \sin x$ and $y = \cos x$, and describe their periodic properties;
TF3.02 - determine, through investigation, using graphing calculators or graphing software, the effect of simple transformations (e.g., translations, reflections, stretches) on the graphs and equations of $y = \sin x$ and $y = \cos x$;
TF3.03 - determine the amplitude, period, phase shift, domain, and range of sinusoidal functions whose equations are given in the form $y = a \sin(kx + d) + c$ or $y = a \cos(kx + d) + c$;
TF3.04 - sketch the graphs of simple sinusoidal functions [e.g., $y = a \sin x$, $y = \cos kx$, $y = \sin(x + d)$, $y = a \cos kx + c$];
TF3.05 - write the equation of a sinusoidal function, given its graph and given its properties;
TF3.06 - sketch the graph of $y = \tan x$; identify the period, domain, and range of the function; and explain the occurrence of asymptotes;
TF4.01 - determine, through investigation, the periodic properties of various models (e.g., the table of values, the graph, the equation) of sinusoidal functions drawn from a variety of applications;
TF4.02 - explain the relationship between the properties of a sinusoidal function and the parameters of its equation, within the context of an application, and over a restricted domain;
TF4.03 - predict the effects on the mathematical model of an application involving sinusoidal functions when the conditions in the application are varied;
TF4.04 - pose and solve problems related to models of sinusoidal functions drawn from a variety of applications, and communicate the solutions with clarity and justifications, using appropriate mathematical forms;
OC2.06 - represent transformations (e.g., translations, reflections, stretches) of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$, using function notation;
OC2.07 - describe, by interpreting function notation, the relationship between the graph of a function and its image under one or more transformations;
OC2.08 - state the domain and range of transformations of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$.

Ontario Catholic School Graduate Expectations
CGE2b - an effective communicator who reads, understands and uses written materials effectively;
CGE3c - a reflective and creative thinker who thinks reflectively and creatively to evaluate situations and solve problems;
CGE3e - a reflective and creative thinker who adopts a holistic approach to life by integrating learning from various subject areas and experience.

Prior Knowledge & Skills
Students should possess a comprehensive knowledge of the concepts introduced and extended throughout this unit.

Teaching/Learning Strategies
- It is intended that this evaluation provide the teacher with a variety of assessment instruments, including a short activity, group work, and individual work. This assessment could, however, take on several other forms. If the teacher desires, this sample evaluation could be used as a unit test, to be completed individually by students. Teachers may wish instead to have pairs or groups of students complete the given tasks and be assessed collectively. Selected activities and questions could even be delivered in the form of an assignment.
- It is recommended that this summative assessment take place over two days to allow for a thorough evaluation of student performance.

Part 1 – Me Tarzan, You Bob

Description

Students measure and determine the equation governing the sinusoidal motion of a swinging pendulum.

Prior Knowledge & Skills

- Students should have some experience in the modelling of sinusoidal equations.
- Students should be able to describe sinusoidal equations and their respective graphs using characteristics such as the domain, range, period, phase shift, vertical translation, and amplitude.

Planning Notes

- Students are paired for this activity, but they produce individual reports.
- The teacher should provide one piece of string of length 50 cm to each group, along with one washer of at least 100 g weight and substantial surface area for use as a bob. The washer should be tied to the end of the string to construct the pendulum.
- A graphing calculator and a motion sensor (such as a CBR) are required for each group, as are measuring tapes or rulers.

Teaching/Learning Strategies

A. Teacher Facilitation

- This activity ties the pendulum demonstration at the beginning of the unit with the modelling of sinusoidal equations.
- Provide each pair of students with a pendulum. For best results, the pendulum should be suspended from a fixed height. Alternatively, one student could hold the string while the other records the data.
- Students record the motion of the bob for three different trials. The motion sensor should be set to collect data for approximately 10 seconds, and the smoothing should be set to *light.*
- There will be some "dead time" at the beginning of the students' plot. The user can reset the domain of the plot to eliminate this space.

B. Student Activity

To aid in the evaluation process, some solutions have been included in italics.

1. Extend the bob 10 cm from the rest position. Start the motion sensor, and set the pendulum in motion. List the characteristics of the resulting graph. Determine its equation. Explain the reasoning by which this equation was arrived at.
2. Repeat this experiment by extending the bob 20 cm and 25 cm from the rest position.
3. Compare and contrast the graphs and their respective equations. *The period will be the same, regardless of the starting position of the bob. The amplitude will increase as the bob is extended further from the vertical. The vertical translation should be constant, because of the fixed position of the ranger. If the domain has been reset on the calculator, the phase shift will also be constant*.
4. Predict with justification the equation that could be used to model the motion of the pendulum if the bob was extended 15 cm from the rest position. Repeat the experiment to confirm the prediction.

Part 2 – Group Assessment

A. Teacher Facilitation

- Each task in this group assessment is preceded by a suggestion of the skill categories that would be most applicable to the given task ([K] indicates Knowledge/Understanding, [I] indicates Thinking/Inquiry/Problem-Solving, [C] indicates Communication, and [A] indicates Application).

B. Student Activity

To aid in the evaluation process, some solutions have been included in italics.

1. The average monthly temperature in New Orleans, Louisiana, is given in the following table:

Month	J	F	M	A	M	J	J	A	S	O	N	D
°C	16.5	18.3	21.8	25.6	29.2	31.9	32.6	32.4	30.3	26.4	21.3	18.0

   a) [K] What is the range of this function?
   b) [K] What is the average yearly temperature?
   c) [K, C] Is this function sinusoidal? Fully explain your answer. *This function is periodic, with a period of 12 months. It is therefore sinusoidal.*
   d) [K] Graph the data using a scatter plot. Does this confirm the answer from question 1C?
   e) [K, A] Given the general sinusoidal function $T(t) = c + a \sin[k(x - s)]$, what do *a, c, k,* and *s* represent?
   f) [K, I] What characteristics of the function correspond to the constants *a, c, k,* and *s*? *The value of c corresponds to the average yearly temperature, the value of a relates to the range of temperatures, k relates to the period (k will have a value of $2\pi \div 12$ in this case), and s corresponds to the phase shift.*
   g) [I] Determine the temperature function **T(t).** [*Hint*: Consider January to be Month 0, February to be Month 1, etc.]

2. In 2001, Windsor, Ontario will receive its maximum amount of sunlight, 15.28 hrs, on June 21, and its least amount of sunlight, 9.08 hrs, on December 21.
   a) [I, A] Due to the earth's revolution about the sun, the hours of daylight function is periodic. Determine an equation that can model the hours of daylight function for Windsor, Ontario.
   b) [K] On what day(s) can Windsor expect 13.5 hours of sunlight?

3. [K, I, A] Tides are cyclical phenomena caused by the gravitational pull of the sun and the moon. On a particular retaining wall, the ocean generally reaches the 3 m mark at high tide. At low tide, the water reaches the 1 m mark. Assume that high tide occurs at 12:00 p.m. and at 12:00 a.m., and that low tide occurs at 6:00 p.m. and 6:00 a.m. What is the height of the water at 10:30 a.m.?

4. The largest Ferris Wheel in the world is the London Eye in England. The height (in metres) of a rider on the London Eye after *t* minutes can be described by the function $h(t) = 70 + 67 \sin \frac{\pi}{15}(t - 30)$.
   a) [A] What is the diameter of this Ferris wheel?
   b) [A, C] Where is the rider at $t = 0$? Explain the significance of this value. *This position indicates the height of the boarding platform.*
   c) [A] How high off the ground is the rider at the top of the wheel?
   d) [A] At what time(s) will the rider be at the bottom of the Ferris wheel?
   e) [A] How long does it take for the Ferris wheel to go through one rotation?

5. At Canada's Wonderland, a thrill seeker can ride the Xtreme Skyflyer. This is essentially a large pendulum of which the rider is the bob. The height of the rider is given for various times:

Time (s)	0	1	2	3	4	5	6	7	8	9
Height (m)	55	53	46	36	25	14	7	5	8	15

   a) [K] Create a graph of the position of the pendulum with respect to the time.
   b) [K, I, A] Find the amplitude, period, vertical translation, and phase shift for this function. [**Note:** *that the table does not follow the bob through one complete cycle, so some thought will be required to answer this question*.]
   c) [I] Determine the equation of the function in the forms $y = a \sin k(x + s) + c$ and $y = a \sin(kx + d) + c$.
   d) [K, C] How could the amplitude be determined without creating the graph or finding the function? *The amplitude is half the range of the data.*

e) [A] What would the rest position of the pendulum be?
f) [A, I] What is the maximum displacement for this pendulum? *The maximum displacement is the range*.
g) [K] The time for one complete cycle is the period. How long would it take to complete 15 cycles?

6. [K, I, A] A mass suspended on a spring will exhibit sinusoidal motion when it moves. If the mass on a spring is 85 cm off the ground at its highest position and 41 cm off the ground at its lowest position and takes 3.0 s to go from the top to the bottom and back again, determine an equation to model the data.

Part 3 – Individual Assessment

A. Teacher Facilitation

Each task in this group assessment is preceded by a suggestion of the skill categories that would be most applicable to the given task ([K] indicates Knowledge/Understanding, [I] indicates Thinking/Inquiry/Problem-Solving, [C] indicates Communication, and [A] indicates Application).

B. Student Activity

1. [K] Graph the function $y = 3 \sin x$, -2π # x # 2π.
2. [K, C] Compare and contrast the characteristics of the graphs of:
   a. $y = \sin x$ and $y = \cos x$ b. $y = \sin x$ and $y = \tan x$ c. $y = \cos x$ and $y = \tan x$
3. [K, C] Does $\sin(-x) = -\sin x$? Explain by describing and sketching their graphs.

4 [K] Given the graph $y = \cos x$, use transformations to sketch the following:
   a. $y = \cos(x - \frac{\pi}{2})$ b. $y = \cos(x + 1)$ c. $y = -\cos 4x$

Assessment & Evaluation of Student Achievement

In this summative assessment, several opportunities exist for the evaluation of all of the knowledge and skill categories. Criteria to be assessed in the activity might include:

- ability to follow the steps outlined in the investigation (Knowledge and Communication);
- ability to compare characteristics of graphs (Communication and Knowledge);
- ability to determine the equation of a sinusoidal function (Inquiry and Knowledge);
- ability to predict results (Application);
- proficiency in calculator usage (Knowledge).

In the group and individual assessments, criteria might include:

- the use of limited information to determine a sinusoidal equation (Inquiry and Knowledge),
- the manipulation of information to answer indirect questions (Application and Communication),
- the proper use of mathematical vocabulary in the justification of conclusions (Communication, Knowledge, and Inquiry),
- graphing techniques (Knowledge and Communication).

All learning skills (initiative, organization, work habits, teamwork, and the ability to work independently) can be evaluated at some point during this summative assessment.

Resources

NRC CNRC – Sunrise/Sunset Tables (http://www.hia.nrc.ca/services/sunmoon/sunmoon.html)

The Herzberg Institute of Astrophysics, a division of the National Research Council of Canada, lists sun and moon data for hundreds of locations across Canada.

World Climate (http://www.worldclimate.com)

This site provides comprehensive climatological data for thousands of locations around the world.